A Geometric Characterization of Extremal Sets in a Hemi-Sphere of S^{∞}.

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Abstract:

The paper deals with unit sphere S^{∞} of a Hilbert space endowed with natural spherical metric. In the paper we give a geometric characterization of extremal sets contained in a hemi-sphere of S^{∞} that generalizes previously known results with respect to the classical Jung theorem.

Key words: *Geometric, Characterization, Extremal Sets, Hemi-Sphere, S^{∞}*

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Introduction

Let (M, d) be a metric space. For a non-empty bounded subset A of M we denote by $d(A)$ its diameter and by $r(A)$ the Chebyshev radius of A, *i.e.* $r(A): -.5em = \inf\{r_c(A): c \in M\}$, where $r_c(A): -.5em = \sup\{d(x,c): x \in A\}$ is the radius of A with respect to a point $c \in M$. A point $c \in M$ is called a Chebyshev center of A if $r_c(A) = r(A)$. For a Banach space $(X, \|.\|)$ with metric naturally generated by its norm $\|.\|$ the Jung constant of X is defined by

$$J(X) = \sup\{\frac{r(A)}{d(A)}: A \text{ is a bounded subset of X with } d(A) > 0\}.$$

A bounded subset A of X with $d(A) > 0$ is said to be a extremal if $r(A) = J(X).d(A)$ (V Nguen-Khac & K Nguen-Van, 2006). Since X possesses a linear structure one sees that in the definition above the supremum can be taken over all subsets A of X with $r(A) = 1$. In the case of a metric space (M, d) without linear structure we consider for $R > 0$ the Jung function $J_M(R)$ of MM defined as follows

$$J_M(R): -.5em = \sup\{\frac{R}{d(A)}: A \subset M \text{ with } r(A) = R\}.$$

A bounded subset A of M with $d(A) > 0$ is said to be R-extremal, if $r(A) = R$ and $J_M(R) = \frac{R}{d(A)}$, and A is said to be extremal, if it is R-extremal for some $R > 0$. For an n-dimensional Euclidean space E^n the Jung theorem asserts (Heinrich Jung, 1899) that

$$J(E^n) = \sqrt{\frac{n}{2(n+1)}}.$$

Furthermore a bounded subset A of E^n is an extremal if and only if A contains all vertices of a regular n-simplex with with edges of length $d(A)$. For a Hilbert space H (the infinite-dimensional case) the Jung constant of H were determined in (NA Routledge, 1952) (*c.f.* (Josef Daneš, 1984)) $J(H) = \frac{1}{\sqrt{2}}$. The main result of (V Nguen-Khac & K Nguen-Van, 2006) gives a geometric characterization extremal sets in a Hilbert space which is an infinite-dimensional version of classical Jung's theorem. Our next aim is to treat the case of S^∞ which is the unit sphere of a Hilbert space endowed the spherical metric. We describe the Jung's function $J_{S^\infty}(R)$ and give a geometric characterization of R-extremal sets in a hemi-sphere of S^∞. The main results of this paper is a generalization of (V Nguen-Khac & K Nguen-Van, 2006) for the case S^∞, which also infinite-dimensional extension of the result of (Boris V Dekster, 1995). It should be note that in (Boris V Dekster, 1995) Dekster gave a version of the classical Jung theorem for unit sphere S^n of $(n+1)$- dimensional Euclidean space with spherical metric. He partially extended his result for Alexandrov spaces of curvature bounded above. Also Lang and Schoreder (Urs Lang & Viktor Schroeder, 1997) gave an upper estimate for the radius of a bounded subset of a $CAT(\kappa)$ space in terms of its diameter. Our main results are Theorem 1.1, Theorem 1.2 and Theorem 1.3 stated below.

Theorem 1.1 *For every $R \in (0, \frac{\pi}{2})$ we have*

$$J_{S^\infty}(R) = \frac{R}{D(R)},$$

where

$$D(R) = 2\sin^{-1}(\frac{\sin R}{\sqrt{2}}).$$

Theorem 1.2 *Let A be a subset of a hemi-sphere of S^∞ with Chebyshev radius $r(A) = R \in (0, \frac{\pi}{2})$ and diameter $d(A) = d$. Then A is R-extremal if and only if for every $\varepsilon \in (0, d)$, for every positive integer mm there exists a subset $A_{\varepsilon,m} = \{x_0, x_1, \dots, x_m\}$ of A_ε such that $d_S(x_i, x_j) > d - \varepsilon$ for all $i \neq j, i, j = 0, 1 \dots, m$, where $d_S(.,.)$ denotes the natural spherical metric.*

Theorem 1.3 *Let A be an extremal set in a hemi-sphere of S^∞ with $r(A) = R$. Then we have $\alpha(A) = D(R)$ and $\chi(A) = R$.* Here $\alpha(A)$ and $\chi(A)$ denote the Kuratowski and Hausdorff measures of non-compactness of AA in metric space (S^∞, d_S) which are defined as inf $\{d > 0 : A$ can be covered by finitely many sets of diameter $\leqslant d\}$ and inf $\{\varepsilon > 0 : A$ can be covered by finitely many balls of radius $\leqslant \varepsilon\}$, respectively. The paper is organized as follows. In §2 we prove several lemmas related to the properties of Chebyshev centers in metric space S^∞ that we shall need in what follows. In particular we note that the existence and uniqueness of Chebyshev centers for subsets of a hemi-sphere of S^∞ are immediate from Proposition 3.1 of (Urs Lang & Viktor Schroeder, 1997). The proof of Theorems 1.1, 1.2 will be given in §3. There we calculate Jung's function $J_{S^\infty}(R)$ via "spherical" technique exposed in §2 and by using our previous result for the case of Hilbert spaces (V Nguen-Khac & K Nguen-Van, 2006). In the final paragraph we give a proof of Theorem 1.3. As a corollary we derive an extension of Gulevich's result for the hemi-sphere case of S^∞.

Notations. Throughout the paper, unless otherwise mentioned, we shall use the following notations.

- For $x, y \in S^\infty$ by $\widehat{xy}$ we mean the geodesic arc joining x to y and by $d_S(x, y)$ the length of $\widehat{xy}$. We also denote by $\overline{xy}$ the linear segment joining x to y and $d(x, y)$ is the distance between x and y in H.

- For each $c \in S^\infty$ we denote by S_c^∞ the hemi-sphere of S^∞ with pole at c, by L_c the closed tangent hyperplane of S^∞ at c. In most of our arguments later L_c will be considered as a Hilbert space with origin at c. By P_c we mean the orthogonal projection S_c^∞ to L_c. The image $P_c(B)$ of a subset B of S_c^∞ under operator P_c is denoted by B'.

- For a subset A of S^{∞} with Chebyshev at c and radius $R \in (0, \frac{\pi}{2})$, by A_{ε} (for each $\varepsilon \in (0, R)$) we denote the set $A \backslash B(c, R-\varepsilon)$ and A'_{ε} denotes the image $P_c(A_{\varepsilon})$ under P_c.

Chebyshev center and Mushroom Lemma

Lemma 2.1 ((Urs Lang & Viktor Schroeder, 1997), Proposition 3.1). *Let A be a subset of a hemi-sphere of S^{∞}, then there exists a unique point $c \in S^{\infty}$ such that $B(c, r(A)) \supset A$.*

Lemma 2.2. *Let A be a subset of a hemi-sphere of S^{∞} and c is the Chebyshev center of A. Then $c \in \overline{co}A'$, where A' is the image of A under P_c and $\overline{co}A'$ denotes the closed convex hull of A' in the tangent space L_c. Proof.* Assume on the contrary that $c \notin \overline{co}A'$. Then by the Hahn- Banach Theorem there exists a closed hyperplane Q of the tangent space L_c separating c from $\overline{co}A'$. Let $c_{1'}$ be the point of Q closest to c and $c_1 \in S_c^{\infty}$ be its preimage under the mapping P_c. For each point $x' \in A'$ let us choose $x \in A$ such that $P_c(x) = x'$ and put

$$r = d_S(c, x), t = d_S(c, c_1), s = d_S(c_1, x), \alpha = \angle(\overline{cx}, \overline{cc_1})$$

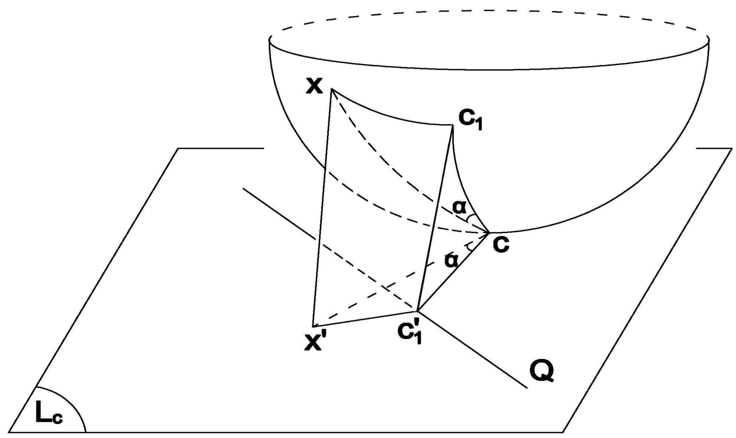

Figure 1

We have (see Fig. 1)

$$d_S(c, x') = \sin r, d_S(c, c_{1'}) = \sin t, \angle(\overline{cx'}, \overline{cc_{1'}}) = \alpha.$$

It is easy to see that

$$d_S(c, x') > d_S(c_{1'}, x'), d_S(c, x') > d_S(c, c_{1'}),$$

or

$$\sin r > s', \sin r > \sin t.$$

Therefore in view of cosine law for spherical triangles cc_1x, cc_1, x' and by an easy computation one gets

$$\cos s > \cos r + \varepsilon,$$

where $\varepsilon = 2\sin^2\frac{t}{2}$ is a positive number independence of x. Clearly

$$s < r - \delta \text{ or } d_S(c_1, x) < d_S(c, x) - \delta$$

for some $\delta > 0$ and independence of x. As x can be taken arbitrarily in A one gets $r_{c_1}(A) < r(A)$. This contradicts to the fact that $r(A)$ is the Chebyshev radius of A. So $c \in \overline{co}A'$. The proof of Lemma 2.2 is completed. Next lemma is a variation of Lemma 2 in (V Nguen-Khac & K Nguen-Van, 2006) and Lemma 4 in (Josef Daneš, 1984).

Lemma 2.3 (Mushroom Lemma). *Let A be a subset of a hemi-sphere of S^∞ with Chebyshev center at c and radius $r(A) = R > 0$. Then for every $\varepsilon \in (0, R)$ we have* (i) $r(A_\varepsilon) = r(A)$; (ii) $c \in \overline{co}A'_\varepsilon$; (iii) *$cc$ is the Chebyshev center of A' in the tangent space L_c. Proof.* (i). Suppose on contrary that $r(A_\varepsilon) < r(A)$. Then c is not Chebyshev center of A_ε. Let c_1 be a Chebyshev center of A_ε then $c_1 \neq c$ and $r_{c_1}(A_\varepsilon) < r_c(A_\varepsilon)$. Denoting by c_2 the mid-point of $\widehat{cc_1}$ we shall prove that

$$r_{c_1}(A_\varepsilon) < r_{c_2}(A_\varepsilon) < r_c(A_\varepsilon). (2.3)$$

Setting

$$r = r_c(A_\varepsilon), r_1 = r_{c_1}(A_\varepsilon), r_2 = r_{c_2}(A_\varepsilon).$$

Since x is arbitrarily taken in A_ε and by the cosine law for spherical triangles xcc_2, xc_1c_2, a direct computation shows that

$$r_1 < r_2 < r,$$

or

$$r_{c_1}(A_\varepsilon) < r_{c_2}(A_\varepsilon) < r_c(A_\varepsilon).$$

Let z be a point strictly inside $\widehat{cc_1}$. We apply the following process of successive divisions in half. Let us first divide the arc $\widehat{cc_1}$ in half, then choose the half-arc containing z, and again divide it in half, and then choose the half-arc containing z, *etc.* until we reach z as a mid-point at some step. By the continuity of $r_x(A_\varepsilon)$ in x one obtains

$$r_{c'}(A_\varepsilon) < r_z(A_\varepsilon) < r_c(A_\varepsilon). (2.4)$$

Now let us fix a point z_0 strictly inside $\widehat{cc_1}$ such that $0 < d_S(z_0, c) < \frac{\varepsilon}{2}$. By (2.4) we have

$$d_S(x, z_0) \leqslant r_{z_0}(A_\varepsilon) < r(A)$$

for $x \in A_\varepsilon$ and

$$d_S(x, z_0) \leqslant d_S(x, c) + d_S(z_0, c) \leqslant r(A) - \varepsilon + \frac{\varepsilon}{2} = r(A) - \frac{\varepsilon}{2}$$

for $x \in A \backslash A_\varepsilon$. Therefore, one can reach that

$$r_{z_0}(A) \leqslant \max\{r_{z_0}(A_\varepsilon), r(A) - \frac{\varepsilon}{2}\} < r(A).$$

This contradicts to the definition of $r(A)$. Hence $r(A_\varepsilon) = r(A)$. (ii) Suppose on contrary that $c \notin \overline{co}A'_\varepsilon$. By the Hahn- Banach Theorem there exists a closed hyperplane Q in tangent space L_c separating c from $\overline{co}A'_\varepsilon$. Let c_1, be the point of Q closest to c and $c_1 \in S_c^\infty$ be its preimage under the mapping P_c, *i.e.* $P_c(c_1) = c_1$, . As in the proof of the Lemma 2.2 one gets

$$r_{c_1}(A_\varepsilon) < r_c(A_\varepsilon) = r(A).$$

Similar as in the proof of (i) above we can find a point $z_0 \neq c$ such that $r_{z_0}(A) < r(A)$. This is a contradiction. Hence $c \in \overline{co}A'_\varepsilon$. (iii) Putting $R' = \sin R$, in view of (ii) we deduce

$$c \in \overline{co}(A' \backslash B(c, R' - \delta)), \forall \delta \in (0, R')$$

and

$$A' \subset B(c, R'),$$

where $\overline{co}(A' \backslash B(c, R' - \delta))$ denotes the closed convex hull of $(A' \backslash B(c, R' - \delta))$ in tangent space L_c. If c is not the Chebyshev center of A', then there exits a point $c_1 \neq c$ in L_c L_c and a positive number $R" < R'$ such that

$$A' \subset B(c_1, R").$$

Let MM be the closed hyperlane in tangent space L_c passing through the mid-point of line segment cc_1 and orthogonal to $c - c_1$. Then M divides L_c into two half-spaces. Let us denote by M_c and M_{c_1} the closed half- spaces containing c and c_1 respectively. For each $x \in L_c$ it is easy to see that

$$x \in M_{c_1}, \text{if } \| x - c_1 \| < \| x - c \|;$$

$$x \in M_c, \text{otherwise}.$$

Therefore

$$B(c_1, R") \backslash B(c, R") \subset M_{c_1}$$

and

$$A' \backslash B(c, R") \subset M_{c_1},$$

since $A' \subset B(c_1, R")$. On the other hand from (ii) one gets

$$c \in \overline{co}(A' \backslash B(c, R''))$$

since $0 < R" < R'$. This implies that $c \in M_{c_1}$ which is a contradiction . So c is the Chebyshev center of A' in L_c. The proof of Lemma 2.3 is completed.

Lemma 2.4. *Let A be a subset of a hemi-sphere of S^∞ with Chebyshev center at c and radius $r(A) = R > 0$. For p, q in A let us denote by $p' = P_c(p)$ and $q' = P_c(q)$. Then* (i) $\| p' - q' \| \leqslant 2sin(\frac{d_S(p,q)}{2})$. (ii) *If $p, q \in A_\varepsilon$ for $\varepsilon \in (0, R)$ then*

$$\| p' - q' \|^2 \geqslant 4\sin^2(\frac{d_S(p,q)}{2}) - 2(\sin^2 R - \sin^2(R - \varepsilon)).$$

The proof of Lemma 2.4 is immediate by using the cosine law for spherical triangles $cpq, cp'q'$, and some elementary inequalities. We shall omit it. Combining Lemmas 2.3 and 2.4 one gets

Lemma 2.5. *Let A be a subset of a hemi-sphere of S^∞ with Chebyshev center at c and radius $r(A) = R > 0$. Then for every $\varepsilon \in (0, R)$ we have* (i) $r(A') = r(A'_\varepsilon) = sinR;$ (ii) $d(A') = d(A'_\varepsilon) = 2sin(\frac{d(A)}{2})$.

Jung's function and Jung's theorem for a hemi-sphere of $S^\infty S^\infty$

Proof of Theorem 1.1. Assume that A is a subset of S^∞ with Chebyshev center at c and radius $r(A) = R \in (0, \frac{\pi}{2})$. Putting $A' = P_c(A)$ one gets $d(A') \geqslant \sqrt{2} r(A')$ (since the Jung constant of a Hilbert space is $\frac{1}{\sqrt{2}}$). By Lemma 2.5

$$d(A) = 2\sin^{-1}(\frac{d(A')}{2}) \geqslant 2\sin^{-1}(\frac{r(A')}{\sqrt{2}}) = 2\sin^{-1}(\frac{\sin R}{\sqrt{2}}) = D(R).$$

Therefore

$$J_{S^\infty}(R) \leqslant \frac{R}{D(R)}.$$

Let $L_c = L + c$ be the closed tangent hyperplane of S^∞ at c. Consider the orthonormal sequence $\{e_n\}_{n=1}^\infty$ in the hyperplane L of Hilbert space H. Putting $u'_n = c + \sin R. e_n \ \forall n \geqslant 1$ we see that sequence $\{u_{n'}\}_{n=1}^\infty$ lie in tangent hyperplane L_c. Let $\{u_n\}_{n=1}^\infty$ be its preimage in hemi-sphere S_c^∞ under P_c *i.e.* $P_c(u_n) = u_{n'}$ $\forall n \geqslant 1$. Clearly

$$d_S(c, u_n) = R, \forall n \geq 1;$$

$$d_S(u_m, u_n) = 2\sin^{-1}(\frac{\sin R}{\sqrt{2}}) \ \forall m, n \geq 1, m \neq n.$$

Let $c_1 \neq c$ be a point in S_c^∞. Setting $c_{1'} = P_c(c_1)$ by the cosine law for spherical triangles $cc_1u_n, cc_{1'}u_n,$ we have

$$\cos(d_S(c_1,u_n)) = \cos(d_S(c,c_1))\cos R + < c_{1'} - c, e_n >. (3.1)$$

Here $<\cdot,\cdot>$ denotes the inner product of L. Setting $A = \{u_n\}_{n=1}^\infty$ and $R_1 = r_{c_1}(A)$, since the sequence $\{e_n\}_{n=1}^\infty$ converges weakly to 0, from (3.1) one gets

$$\cos R_1 < \cos R \ (\Rightarrow R_1 > R).$$

So $r(A) = r_c(A) = R$ and

$$d(A) = 2\sin^{-1}(\frac{\sin R}{\sqrt{2}}).$$

We conclude that $J_{S^\infty}(R) = \frac{R}{D(R)}$ and the proof of Theorem 1.1 is completed.

Proof of Theorem 1.2. Assume that A is an extremal set in S^∞ with Chebyshev center at c and radius $r(A) = R \in (0,\pi/2)$. Then we have $d(A) = D(R) = 2\sin^{-1}(\frac{\sin R}{\sqrt{2}})$. By Lemma 2.5 one has for each $\varepsilon \in (0,R)$

$$r(A') = r(A'_\varepsilon) = \sin R;$$

$$d(A') = d(A'_\varepsilon) = 2\sin(\frac{d(A)}{2}) = \sqrt{2}\sin R.$$

Hence A'_ε is an extremal set in the Hilbert space L_c. We now apply the main result of (V Nguen-Khac & K Nguen-Van, 2006) to A'_ε for every $\delta \in (0, r(A'_\varepsilon))$, for any positive integer number m there exists $x_{0'}, x_{1'}, \ldots, x_{m'} \in A'_\varepsilon$ such that

$$\| x_{i'} - x_{j'} \| > d(A') - \sqrt{2}\delta = \sqrt{2}(\sin R - \delta) \ \forall i,j = 0,1,\ldots,m, i \neq j.$$

Let us choose $x_0, x_1, \ldots, x_m \in A_\varepsilon$ such that $P_c(x_i) = x_{i'} \ \forall i = 0,1,\ldots m$. In view of Lemma 2.4

$$d_S(x_i,x_j) \geqslant 2\sin^{-1}(\frac{\| x_{i'} - x_{j'} \|}{2}) \geqslant 2\sin^{-1}(\frac{(\sin R - \delta)}{\sqrt{2}}) \ \forall i,j = 0,1,\ldots,m, i \neq j.$$

Let $\delta > 0\delta > 0$ $2\sin^{-1}(\frac{(\sin R - \delta)}{\sqrt{2}}) > 2\sin^{-1}(\frac{\sin R}{\sqrt{2}}) - \varepsilon = d(A) - \varepsilon.$

Putting

$$A_{\varepsilon,m}: -.5em = \{x_0, x_1, \ldots, x_m\}$$

we see that this is clearly a subset of $A_\varepsilon A_\varepsilon$ satisfying

$$d_S(x_i,x_j) > d(A) - \varepsilon, \forall i,j = 0,1\ldots,m, i \neq j.$$

Conversely assume that for every $\varepsilon \in (0, R)$, for any positive integer number m there exists a subset $A_{\varepsilon,m} = \{x_0, x_1, \dots, x_m\} \subset A_\varepsilon$ such that: $d_S(x_i, x_j) > d(A) - \varepsilon$ for all $i, j = 0,1\dots, m, i \neq j$. Putting $x_{i'} = P_c(x_i)$ $\forall i = 0,1\dots, m$, from Lemma 2.4 one deduces

$$\| x_{i'} - x_{j'} \|^2 \geqslant 4\sin^2(\frac{d(A) - \varepsilon}{2}) - 2(\sin^2 R - \sin^2(R - \varepsilon)), \forall i, j = 0,1\dots, m, i \neq j.$$

For each $\delta \in (0, d(A'))$ we choose $\varepsilon \in (0, R)$ sufficiently small so that

$$4\sin^2(\frac{d(A) - \varepsilon}{2}) - 2(\sin^2 R - \sin^2(R - \varepsilon)) > (2\sin(\frac{d(A)}{2}) - \delta)^2.$$

Then

$$\| x_{i'} - x_{j'} \| > 2\sin(\frac{d(A)}{2}) - \delta = d(A') - \delta, \forall i, j = 0,1\dots, m, i \neq j.$$

Hence by the main result of (V Nguen-Khac & K Nguen-Van, 2006) A' is an extremal set in Hilbert space L. Therefore

$$d(A') = \sqrt{2} r(A') = \sqrt{2}\sin R.$$

Thus by Lemma 2.5 one can achieve that

$$d(A) = 2\sin^{-1}(\frac{d(A')}{2}) = 2\sin^{-1}(\frac{\sin R}{\sqrt{2}}) = D(R).$$

So AA is R-extremal set in S^∞. The proof of the Theorem 1.2 is completed.

Measures of non-compactness of extremal sets

Proof of Theorem 1.3. First we prove that $\alpha(A) = D(R)$. Obviously $\alpha(A) \leqslant D(R)$ since $d(A) = D(R)$. Assume on the contrary that $\alpha(A) < D(R)$. Then one can choose $\varepsilon > 0$ satisfying $\alpha(A) < D(R) - \varepsilon$, and so subsets $A_1, A_2, \dots, A_m$ of A such that

$$A = \bigcup_{i=1}^{m} A_i,$$

$$d(A_i) \leqslant D(R) - \varepsilon, \forall i = 1,2, \dots, m.$$

By Theorem 1.2 there exists a subset

$$A_\varepsilon = \{x_0, x_1, \dots, x_m\} \subset A$$

such that

$$d_S(x_i, x_j) > D(R) - \varepsilon, \forall i, j \in \{0,1,\dots,m\}, i \neq j.$$

Clearly there exists at least one set among $A_1, A_2, \dots, A_m$, say A_1, such that A_1 consisting at least two points of A_ε. Hence $d(A_1) > D(R) - \varepsilon$. This contradicts to the choice of $A_1, A_2, \dots, A_m$ above. So $\alpha(A) = D(R)$. Now we show that $\chi(A) = R$. Clearly $\chi(A) \leqslant R$. Let R_1 be a number satisfying $R_1 > \chi(A)$. Then there exist a finite number of balls of radius R_1 say $B_1, B_2, \dots, B_m$ of S^∞ such that

$$A \subset \bigcup_{i=1}^{m} B_i .$$

By Theorem 1.2 for each positive integer number nn there exists a subset C_n consisting $n.m + 1$ points of A such that the distance between arbitrary two points of C_n not less than $d(A) - \frac{1}{n}$. Denoting by

$$J_i = \{n \in \mathbb{N}: B_i \text{ consisting at least } n + 1 \text{ pointof } C_n\}, i = 1,2,\dots,m.$$

one sees that there exists at least one set among $J_1, J_2, \dots, J_m$, say J_1 such that $|J_1| = \infty$, where $|J_1|$ denotes the cardinality of J_1. Now for each number $\varepsilon \in (0, R)$ and each positive integer number k, we choose a positive integer number $n \in J_1$ sufficiently large so that $n \geqslant k$ and $\frac{1}{n} < \varepsilon$. Then $A \cap B_1$ contains $k + 1$ points of A satisfying the property that distance between any two points of which are not less than $d(A) - \varepsilon$. This implies

$$d(A \cap B_1) = d(A) = D(R),$$

so $A \cap B_1$ is also an extremal set in S^∞ by Theorem 1.2. Hence

$$2\sin^{-1}(\frac{\sin R}{\sqrt{2}}) = d(A) = d(A \cap B_1) = 2\sin^{-1}(\frac{\sin(r(A \cap B_1))}{\sqrt{2}}).$$

It follows that $r(A \cap B_1) = R$, so $R_1 \geqslant R$. Hence $\chi(A) \geqslant R$ and one concludes $\chi(A) = R$. The proof of Theorem 1.3 is completed.

From Theorem 1.3 one deduces that there are no relatively compact sets in a hemi-sphere of $S^\infty S^\infty$ which are extremal. In fact we obtain the following extension of Gulevich's result in (NM Gulevich, 1990) for our case.

Corollary 4.1. *Suppose that AA is a relatively compact set in a hemi-sphere of S^∞ with $d(A) > 0$. Then $r(A) < \frac{R}{D(R)} d(A)$, where $D(R)$ is as above.*

Remark 4.2. (i) For any bounded subset A of S^∞ with radius $r(A) = R \in (0, \frac{\pi}{2})$ we can check easily that the following inequality holds

$$\chi(A) \leqslant J_{S^\infty}(R)\alpha(A) (4.1)$$

Indeed if $d > \alpha(A)$ one can be choose subsets A_1, A_2,..., A_m of A such that

$$A = \bigcup_{i=1}^{m} A_i,$$
$$d(A_i) \leqslant d, \forall i = 1,2,\dots,m.$$

Setting

$$R_1 = \max\{r(A_i): i = 1,2\dots,m\} = r(A_{i_0})$$

for some $i_0 \in \{1,2,\dots,m\}$ we have

$$R \geqslant R_1 \geqslant \chi(A),$$

and

$$d \geqslant d(A_{i_0}).$$

Therefore

$$\chi(A) \leqslant R_1 \leqslant J_{S^\infty}(R_1)d(A_{i_0}) \leqslant J_{S^\infty}(R)d.$$

Since d is a arbitrarily greater than $\alpha(A)$ one gets (4.1). (ii) In view of Theorem 1.3 we see that the inequality in (4.1) holds if A is an extremal set.

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