A New Inertial Subgradient Projection Algorithm for Solving Pseudomonotone Equilibrium Problems

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Abstract:

In this paper, we introduce a new inertial subgradient projection algorithm for finding a solution of an equilibrium problem in a real Hilbert space. The proposed algorithm combines subgradient projection methods with the self-adaptive and inertial techniques to generate iteration sequences. The convergent theorem are established under mild assumptions. Several fundamental experiments are shown to illustrate our algorithm.

Keywords: *equilibrium problem, subgradient, projection, pseudomonotone, self adaptive stepsize, inertial technique*

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Introduction

Let H denote a real Hilbert space and C be a nonempty closed convex subset of H. The paper is interested in a method for solving the following equilibrium problem (EPs):

Find$x^* \in C$ suchthat$f(x^*, y) \geq 0 \quad \forall y \in C$,

with $f: \mathrm{H} \times \mathrm{H} \to \mathbb{R}$ being an equilibrium bifunction, i.e., $f(x, x) = 0$ for every $x \in \mathrm{H}$ and $f(x, \cdot)$ being convex and subdifferentiable on H for every fixed point $x \in \mathrm{H}$. Throughout this article, we denote $Sol(EPs)$ the solution set of Problem. Problem (EPs) was first presented in convex game models in 1955 (Anh et al., 2005). After the appearance of the paper by Blum and Oettli (Blum 1994), the Problem (EPs) contains as special cases some classes of optimization, Nash equilibria, Kakutani fixed points and variational inequality and some others as special cases (see e.g. (Blum, 1994; Konnov, 2001)). Some approaches have been proposed to solve Problem (EPs) in both finite and infinite dimensional spaces, can be found, for example, in (Anh et al., 2005; Hung, 2011; Van Thang, 2022; Van Thang & Khoa, 2022). Among these methods, the projection method is one of the most important methods and are widely used. In general, the projection algorithm is not convergent even for the monotone variational inequality, which is a special case of the monotone equilibrium problem. In (Korpelevich, 1977), the extragradient algorithm for a monotone variational inequality first introduced by Korpelevich, which is defined by:

$$\begin{cases} x^0 \in C, \\ y^n = \Pi_C(x^n - \lambda_n F(x^n)), \\ x^{n+1} = \Pi_C(x^n - \lambda_n F(y^n)), \end{cases}$$

where $\lambda_n \in (0, \frac{1}{L})$, L is the Lipschitz constant of cost mapping F and Π_C is metric projection from H onto C. Afterward, Korpelevich's extragradient method has been improved and extended to variational inequality and equilibrium problems in different ways (Konnov, 2001; Oggioni et al., 2012). Recently, the algorithms with inertial steps (inertial-type algorithm) have received a lot of research attention from the authors. Inspired by the recent trend of inertial extrapolation methods for solving variational inequality and equilibrium problems, our aim in this paper is to modify the existing algorithm in paper (Van Thang, 2022) to develop a new algorithms for solving a pseudomonotone equilibrium problem with a Lipschitz type condition. The algorithm is designed by combining the subgradient projection method with the self-adaptive and inertial techniques. We establish the convergence result of the iteration sequence generated by the proposed algorithm under mild assumptions.

The remainder of this paper is organized as follows: We present some definitions and preliminary results for further use in Section 2. In Section 3, we present an inertial subgradient projection algorithm for solving Problem (EPs) and prove the convergence theorems of the algorithm. In the last section, several fundamental experiments are provided to illustrate the proposed algorithm.

Preliminaries

In this section, we recall some necessary concepts, lemmas that will be used in proving the main results of the paper.

Denote $\rho(A,B)$ by the Hausdorff distance between two set A and B, that is: $\rho(A,B) := \max\{d(A,B), d(B,A)\}$, $\qquad d(A,B) := \sup_{a\in A}\inf_{b\in B} \| a-b \|$.

Definition 1 *(Definition 2.1.5, (Konnov, 2001))* Let C be a nonempty convex subset of a real Hilbert space H. A bifunction $f: C\times C \to \mathbb{R}$ is said to be

(a) *η-strongly monotone on C*, if $f(x,y)+f(y,x) \le -\eta \| x-y \|^2 \quad \forall x,y\in C$;

(b) *monotone on C*, if $f(x,y)+f(y,x)\le 0 \quad \forall x,y\in C$;

(c) *pseudomonotone on C*, if $f(x,y)\ge 0 \Rightarrow f(y,x)\le 0 \ \ \forall x,y\in C$.

Let $\{x^k\}$ be an arbitrary sequence in H, we denote by $x^k\to p$ the strong convergence of $\{x^k\}$ to p and $x^k \rightharpoonup p$ the weak convergence. We now recall some weak continuity concepts of a function.

Definition 2 [Definition 2.1, (Khoa & Van Thang, 2022)] A mapping g: H $\to (-\infty,+\infty]$ is called to be

(a) *sequentially weakly continuous* on H, if $\{x^k\}\subset$ H converges weakly to $\bar{x}\in$ H, it follows that $\lim_{k\to\infty} g(x^k) = g(\bar{x})$;

(b) *sequentially weakly lower semicontinuous at $\bar{x}$*, if $\liminf_{x\rightharpoonup\bar{x}} g(x)\ge g(\bar{x})$, and sequentially weakly lower semicontinuous on H if this holds for every $\bar{x}$ in H.

(c) *sequentially weakly upper semicontinuous at $\bar{x}$*, if $\limsup_{x\rightharpoonup\bar{x}} g(x)\le g(\bar{x})$, and sequentially weakly upper semicontinuous on H if this holds for every $\bar{x}$ in H.

Definition 3 *(Definition 1.1.3, (Konnov, 2001)) Let C be a nonempty convex set in H. A function $g: C\to\mathbb{R}\cup\{+\infty\}$ is said to be convex on C if for each pair of points $x,y\in C$ and for all $\lambda\in[0,1]$, we have*

$$g(\lambda x+(1-\lambda)y)\le \lambda g(x)+(1-\lambda)g(y).$$

Definition 4 *(see (Konnov, 2001))* Let C be a nonempty closed convex subset in H. The metric projection from H onto C is defined by Π_C and

$$\Pi_C(x) = \operatorname{argmin}\{\| x - y \| : \ y \in C\}, \forall x \in \mathrm{H}.$$

From the definition, it is easy to see that Π_C has the following characteristic properties.

Lemma 5 *(Proposition 1.2.1, (Konnov, 2001))*

(a) For any $x \in \mathrm{H}, z = \Pi_C(x)$ if and only if $\langle z - x, y - z\rangle \le 0, \quad \forall y \in C$

(b) $\| \Pi_C(x) - \Pi_C(y) \| \le \| x - y \|, \quad \forall x, y \in \mathrm{H};$

(c) $\| \Pi_C(x) - z \|^2 \le \| x - z \|^2 - \| \Pi_C(x) - x \|^2, \quad \forall x \in \mathrm{H}, z \in C.$

Lemma 6 *(Lemma 2.5, (Van Thang, 2022)) For every $x,\ y,\ z \in H$ and $\xi \in \mathbb{R}$, the following inequality holds*

$$\| \xi x + (1 - \xi) y \|^2 = \xi \| x \|^2 + (1 - \xi) \| y \|^2 - \xi(1 - \xi) \| x - y \|^2.$$

Lemma 7 *(Lemma 2.6, (Van Thang, 2022)) For every $x,\ y \in H$, We have the following assertions.*

(a) $\| x + y \|^2 = \| x \|^2 + 2\langle x, y\rangle + \| y \|^2;$

(b) $\| x + y \|^2 \le \| x \|^2 + 2\langle y, x + y\rangle.$

The subdifferential of a convex function $g: C \to \mathbb{R} \cup \{+\infty\} g: C \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\partial g(x) = \{a \in \mathrm{H}: \langle a, y - x\rangle \le g(y) - g(x) \quad \forall y \in C\}.$$

In convex programming, we have the following result.

Lemma 8 [Theorem 27.4, (Tyrrell Rockafellar, 1970) *Let C be a convex subset of a real Hilbert space H and $g: C \to \mathbb{R} \cup \{+\infty\}$ be subdifferentiable. Then, x^* is a solution to the following convex problem:* $\min\{g(x): x \in C\}$

if and only if $0 \in \partial g(x^*) + N_C(x^*)$, where $N_C(x^*)$ is the outer normal cone of C at $x^* \in C$, that is, $N_C(x^*) = \{a \in \mathrm{H}: \langle a, y - x^*\rangle \le 0 \quad \forall y \in C\}$.

Lemma 9 [Lemma 2.8, (Van Thang, 2022)] *Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences of nonnegative real numbers satisfying two following conditions:*

(a) $a_{n+1} \le a_n + c_n(a_n - a_{n-1}) + b_n, \quad \forall k \ge 1, \sum_{n=1}^{+\infty} b_n < \infty$

(b) there exists a real number c such that $0 \le c_n \le c < 1$ for all $k \ge 1$.

Then,

(a) $\sum_{n=1}^{\infty} [a_n - a_{n-1}]_+ < \infty$, where $[a_n - a_{n-1}]_+ := \max\{a_n - a_{n-1}, 0\}$

(b) there exists $a^* \in [0, +\infty)$ such that $\lim_{n\to\infty} a_n = a^*$.

Lemma 10 [Lemma 2.39, (Bauschke & Combettes, 2011)] *Let C be a nonempty subset in real Hilbert space H and $\{x^n\} \subset H$ satisfy the following conditions:*

(a) for all $x \in C$, $\lim_{n\to+\infty} \| x^n - x \|$ exists;

(b) every sequentially weak cluster point of $\{x^n\}$ is in C.

Then, the sequence $\{x^n\}$ converges weakly to a point in C.

Inertial subgradient projection algorithm

In this section, we introduce a new inertial algorithm for finding a solution of the (EPs) and show its weak convergence. It is described as follows.

In order to find a solution of (EPs), we assume that the bifunction $f: \mathrm{H} \times \mathrm{H} \to \mathbb{R}$ satisfies the following conditions:

(a) $f(x, y)$ is pseudomonotone on $\mathrm{H} \times \mathrm{H}$ and $f(\cdot, y)$ is sequentially weakly upper semicontinuous on H;

(b) there exists a real positive number L such that

$\rho(\partial_2 f(x,\cdot)(x), \partial_2 f(y,\cdot)(y)) \leq L \| x - y \|, \ \forall x \in \mathrm{H}, y \in C,$

where $\partial_2 f(x,\cdot)(x)$ is subdifferential of $f(x,\cdot)$ at x, i.e.,

$\partial_2 f(x,\cdot)(x) = \{a \in \mathrm{H}: \ \langle a, z - y \rangle \leq f(x, z), \ \forall z \in C;$

(a) $Sol(EPs)$ is nonempty.

(b) $f(x,\cdot)$ is convex and subdifferentiable on H;

Algorithm 1 *Choose starting points*

$x^0, \ x^1 \in Hx^0, \ x^1 \in H, \ \xi_0 > 0, \ L^* > L, \ \alpha \in (0,1), \ \gamma \in (0,2)$

$\xi_0 > 0, \ L^* > L, \ \alpha \in (0,1), \ \gamma \in (0,2)$ *and the positive sequences* $\{\theta_n\}$, $\{\mu_n\}$, $\{\kappa_n\}$ *satisfying*

$$\begin{cases} 0 < \theta_n, & \sum_{n=0}^{+\infty} \theta_n < +\infty, \mu_n \in (0,1), \quad \sum_{n=0}^{+\infty} \mu_n < +\infty \\ 0 < \kappa_n, & \lim_{n \to +\infty} \kappa_n = 0. \end{cases} \tag{1}$$

Set $n = 1$ and go to Step 1.

Step 1. Choose α_n such that $0 \leq \alpha_n \leq \overline{\alpha}_n$

$$\bar{\alpha}_n = \begin{cases} \min\{\frac{\mu_n}{\| x^n - x^{n-1} \|^2}, \alpha\}, & if \quad x^n \neq x^{n-1}, \\ \alpha & \text{otherwise.} \end{cases} \tag{2}$$

Step 2. Compute

$$w^n = x^n + \alpha_n (x^n - x^{n-1}). \tag{3}$$

Step 3. Choose $U_{w^n} \in \partial_2 f(w^n, w^n)$. Find $y^n \in C$ such that

$$y^n = \Pi_C (w^n - \xi_n U_{w^n}), \tag{4}$$

If $y^n = w^n$ then Stop. Otherwise, go to the next step.

Step 4. Take $U_{y^n} \in B(U_{w^n}, L^* \| w^n - y^n \|) \cap \partial_2 f(y^n, y^n)$, where

$B(U_{w^n}, L^* \| w^n - y^n \|) := \{x \in H : \| x - U_{w^n} \| \leq L^* \| w^n - y^n \|\}$. Set

$$\Xi^n = w^n - y^n - \xi_n (U_{w^n} - U_{y^n}). \tag{5}$$

Compute

$$x^{n+1} = \Pi_C (w^n - \tau_n \xi_n U_{y^n}) \tag{6}$$

and

$$\xi_{n+1} = \begin{cases} \min\left\{\frac{\nu \| x^n - y^n \|}{\| U_{w^n} - U_{y^n} \|}, \xi_n + \theta_n\right\}, & if \;\; U_{w^n} - U_{y^n} \neq 0, \\ \xi_n + \theta_n & \text{otherwise,} \end{cases} \tag{7}$$

where τ_n is defined by

$$\tau_n = \begin{cases} (\gamma + \kappa_n) \frac{\langle w^n - y^n, \Xi^n \rangle}{\| \Xi^n \|^2}, & if \quad \| \Xi^n \| \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Step 5. Let $n = n + 1$ and return to *Step 1.*

Remark 2 *(i) If $y^n = w^n$, it follows from (3.4) that $w^n = \Pi_C (w^n - \xi_n U_{w^n})$, which implies that*

$$0 \leq \langle U_{w^n}, x - w^n \rangle \leq f(w^n, x), \ \forall x \in C.$$

So, w^n is a solution of the (EPs).

(ii) In (Van Thang, 2022), we have developed a

Lemma 3 *If the assumptions* $(M_1) - (M_4)$ *hold,* $x^* \in Sol(EPs)$ *and the sequences* $\{x^n\}, \{w^n\}, \{y^n\}$ *are generated by Algorithm 3.1. Then*

$\lim_{n\to\infty} \parallel w^n - x^n \parallel = \lim_{n\to\infty} \alpha_n \parallel x^n - x^{n-1} \parallel = 0;$

$\xi_n \in [\min\{\frac{\nu}{L^*}, \xi_0\},\ \xi_0 + \Theta],\ \forall k \geq 0$ and $\lim_{n\to\infty} \xi_n = \xi$, where $\sum_{n=0}^{+\infty} \theta_n = \Theta$;

$\parallel x^{n+1} - x^* \parallel^2 \leq \parallel w^n - x^* \parallel^2 - (\gamma + \kappa_n)(2 - \gamma - \kappa_n)\dfrac{(\xi_{n+1} - \nu\xi_n)^2}{(\xi_{n+1} + \nu\xi_n)^2} \parallel w^n - y^n \parallel^2.$

Proof. We have from (2) that $\alpha_n \parallel x^n - x^{n-1} \parallel^2 \leq \mu_n$, which together with Condition (1) implies that

$\sum_{n=1}^{\infty} \alpha_n \parallel x^n - x^{n-1} \parallel^2 < \infty.$

This together with (3) implies $\lim_{n\to\infty} \parallel w^n - x^n \parallel = \lim_{n\to\infty} \alpha_n \parallel x^n - x^{n-1} \parallel = 0.$

Now, we prove (b). Let $U_{w^n} - U_{y^n} \neq 0$. Then, we deduce from the assumption (M_2) that

$$\frac{\nu \parallel w^n - y^n \parallel}{\parallel U_{w^n} - U_{y^n} \parallel} \geq \frac{\nu \parallel w^n - y^n \parallel}{L^* \parallel w^n - y^n \parallel} = \frac{\nu}{L^*}. \tag{9}$$

By (7) and using mathematical induction proof method, it is easy to see that $\{\xi_n\}$ belongs to $[\min\{\frac{\nu}{L^*}, \xi_0\},\ \xi_0 + \Theta],\ \forall k \geq 0$ Set $(\xi_{n+1} - \xi_n)^+ = \max\{0, \xi_{n+1} - \xi_n\}$ and

$(\xi_{n+1} - \xi_n)^- = \max\{0, -(\xi_{n+1} - \xi_n)\}.$

From (7), we get

$$\sum_{n=0}^{+\infty} (\xi_{n+1} - \xi_n)^+ \leq \sum_{n=0}^{+\infty} \tau_n < +\infty. \tag{10}$$

Assume that $\sum_{n=0}^{+\infty} (\xi_{n+1} - \xi_n)^- = +\infty.$

Using the following quality

$$\xi_{n+1} - \xi_n = (\xi_{n+1} - \xi_n)^+ - (\xi_{n+1} - \xi_n)^-,$$

we obtain

$$\xi_{n+1} - \xi_0 = \sum_{k=0}^{n} (\xi_{k+1} - \xi_k) = \sum_{k=0}^{n} (\xi_{k+1} - \xi_k)^+ - \sum_{k=0}^{n} (\xi_{k+1} - \xi_k)^-. \quad (11)$$

Taking the limit as $n \to \infty$ on both sides of the above inequality, using (10), one has $\xi_n \to -\infty$, which is a contradiction. Therefore, $\sum_{n=0}^{+\infty} (\xi_{n+1} - \xi_n)^- < +\infty$. Which together with (10) implies that $\lim_{n\to\infty}\xi_n = \xi \in [\min\{\frac{\nu}{L}, \xi_0\}, \ \xi_0 + \Theta]$.

Next, we prove (c). By using $x^* \in Sol(EPs)$, $f(y^n, y^n) = 0$, $U_{y^n} \in \partial_2 f(y^n, y^n)$ and the pseudomonotone assumption of f, one has

$$\tau_n\xi_n\langle U_{y^n}, y^n - x^*\rangle \geq \tau_n\xi_n[f(y^n, y^n) - f(y^n, x^*)] \geq 0. \quad (12)$$

From (6) and Lemma 5 (c), it follows that

$$\| x^{n+1} - x^* \|^2 = \| \Pi_C(w^n - \tau_n\xi_n U_{y^n}) - x^* \|$$

$$\leq \| w^n - \tau_n\xi_n U_{y^n} - x^* \|^2 - \| w^n - \tau_n\xi_n U_{y^n} - x^{n+1} \|^2$$

$$= \| w^n - x^* \|^2 - 2\tau_n\xi_n\langle w^n - x^*, U_{y^n}\rangle + 2\tau_n\xi_n\langle w^n - x^{n+1}, U_{y^n}\rangle - \| x^{n+1} - w^n \|^2$$

$$= \| w^n - x^* \|^2 - \| x^{n+1} - w^n \|^2 - 2\tau_n\xi_n\langle y^n - x^*, U_{y^n}\rangle + 2\tau_n\xi_n\langle y^n - x^{n+1}, U_{y^n}\rangle.$$

Combining this and (12), we have

$$\| x^{n+1} - x^* \| \leq \| w^n - x^* \|^2 - \| x^{n+1} - w^n \|^2 + 2\tau_n\xi_n\langle y^n - x^{n+1}, U_{y^n}\rangle. \quad (13)$$

By Step 3, one has

$$0 \leq 2\tau_n\langle w^n - y^n - \xi_n U_{w^n}, y^n - x\rangle, \ \forall x \in C,$$

which implies that

$$0 \leq 2\tau_n\langle w^n - y^n - \xi_n U_{w^n}, y^n - x^{n+1}\rangle. \quad (14)$$

It follows from (13) and (14) that

$$\| x^{n+1} - x^* \|^2 \leq \| w^n - x^* \|^2 - \| x^{n+1} - w^n \|^2 + 2\tau_n\xi_n\langle y^n - x^{n+1}, U_{y^n}\rangle + 2\tau_n\langle w^n - y^n - \xi_n U_{w^n}, y^n - x^{n+1}\rangle$$

$$= \| w^n - x^* \|^2 - \| w^n - x^{n+1} - \tau_n\Xi^n \|^2 + (\tau_n \| \Xi^n \|)^2 - 2\tau_n\langle w^n - x^{n+1}, \Xi^n\rangle + 2\tau_n\langle y^n - x^{n+1}, \Xi^n\rangle$$

$$= \| w^n - x^* \|^2 - \| w^n - x^{n+1} - \tau_n\Xi^n \|^2 + (\tau_n \| \Xi^n \|)^2 + 2\tau_n\langle y^n - w^n, \Xi^n\rangle \leq \| w^n - x^* \|^2 + (\tau_n \| \Xi^n \|)^2 + 2\tau_n\langle y^n - w^n, \Xi^n\rangle.$$

We deduce from the above inequality and (8) that

$$
\begin{aligned}
&\| x^{n+1} - x^* \|^2 \\
&\leq \| w^n - x^* \|^2 + (\gamma + \kappa_n)^2 \left(\frac{\langle w^n - y^n, \Xi^n \rangle}{\| \Xi^n \|^2} \right)^2 \\
&\qquad \| \Xi^n \|^2 + 2(\gamma + \kappa_n) \frac{\langle w^n - y^n, \Xi^n \rangle}{\| \Xi^n \|^2} \langle y^n - w^n, \Xi^n \rangle \\
&\leq \| w^n - x^* \|^2 - (\gamma + \kappa_n)(2 - \gamma - \kappa_n) \frac{(\langle w^n - y^n, \Xi^n \rangle)^2}{\| \Xi^n \|^2}.
\end{aligned} \tag{15}
$$

On the other hand, using (5) and (7), one has

$$
\begin{aligned}
\langle w^n - y^n, \Xi^n \rangle &= \langle w^n - y^n, w^n - y^n - \xi_n (U_{w^n} - U_{y^n}) \rangle \\
&= \| w^n - y^n \|^2 - \xi_n \langle w^n - y^n, U_{w^n} - U_{y^n} \rangle \\
&\geq \| w^n - y^n \|^2 - \xi_n \| w^n - y^n \| \| U_{w^n} - U_{y^n} \| \\
&\geq (1 - \frac{\nu \xi_n}{\xi_{n+1}}) \| w^n - y^n \|^2.
\end{aligned} \tag{16}
$$

From the definition of Ξ^n and Step 3, we obtain

$$
\| \Xi^n \| = \| w^n - y^n - \xi_n (U_{w^n} - U_{y^n}) \| \leq \| w^n - y^n \| + \xi_n \| U_{w^n} - U_{y^n} \|
$$

$$
\leq (1 + \frac{\nu \xi_n}{\xi_{n+1}}) \| w^n - y^n \|,
$$

which together with (16) implies that

$$
\langle w^n - y^n, \Xi^n \rangle \geq (1 - \frac{\nu \xi_n}{\xi_{n+1}}) \| w^n - y^n \|^2 \geq \xi_{n+1} \frac{\xi_{n+1} - \nu \xi_n}{(\xi_{n+1} + \nu \xi_n)^2} \| \Xi^n \|^2. \tag{17}
$$

$$
\| x^{n+1} - x^* \|^2 \leq \| w^n - x^* \|^2 - (\gamma + \kappa_n)(2 - \gamma - \kappa_n) \xi_{n+1} \frac{\xi_{n+1} - \nu \xi_n}{(\xi_{n+1} + \nu \xi_n)^2} \langle w^n - y^n, \Xi^n \rangle
$$

$$
\leq \| w^n - x^* \|^2 - (\gamma + \kappa_n)(2 - \gamma - \kappa_n) \frac{(\xi_{n+1} - \nu \xi_n)^2}{(\xi_{n+1} + \nu \xi_n)^2} \| w^n - y^n \|^2.
$$

Lemma 4 *Assume that the assumptions $(M_1) - (M_4)$ holds and $x^* \in Sol(EPs)$. Then, we have the following assertions*

(a) the $\lim_{n\to\infty} \| x^n - x^* \|^2$ exists

(b) the sequences $\{x^n\}$ and $\{w^n\}$ are bounded;

(c) $\lim\limits_{n\to\infty} [\| x^n - x^* \|^2 - \| x^{n-1} - x^* \|^2]_+ = 0$, where

$[a]_+ := \max\{a, 0\}$ for each $a \in \mathbb{R}$.

Proof. Let $x^* \in Sol(EPs)$. By Lemma 6 and (3), one has

$$\| w^n - x^* \|^2 = \| x^n - \alpha_n(x^n - x^{n-1}) - x^* \|^2$$

$$= \| (1+\alpha_n)(x^n - x^*) + (-\alpha_n)(x^{n-1} - x^*) \|^2 \tag{18}$$

$$= (1+\alpha_n) \| x^n - x^* \|^2 + (-\alpha_n) \| x^{n-1} - x^* \|^2 + \alpha_n(1+\alpha_n) \| x^n - x^{n-1} \|^2.$$

We have from 3 (b) that

$$\| x^{n+1} - x^* \|^2 \leq \| w^n - x^* \|^2, \ \forall k,$$

which together with (18) implies that, for all k

$$\| x^{n+1} - x^* \|^2 \leq (1+\alpha_n) \| x^n - x^* \|^2 + (-\alpha_n) \| x^{n-1} - x^* \|^2 + \alpha_n(1+\alpha_n) \| x^n - x^{n-1} \|^2 \tag{19}$$

$$= \| x^n - x^* \|^2 + \alpha_n(\| x^n - x^* \|^2 - \| x^{n-1} - x^* \|^2) + \alpha_n(1+\alpha) \| x^n - x^{n-1} \|^2.$$

Letting $a_n = \| x^n - x^* \|^2, c_n = \alpha_n(1+\alpha) \| x^n - x^{n-1} \|^2$, then by Lemma 3 (a), we get

$$\sum_{n=1}^{\infty} \alpha_n(1+\alpha) \| x^n - x^{n-1} \|^2 < +\infty.$$

By Lemma 9, we can conclude that $\lim_{n\to\infty} \| x^n - x^* \|^2$ exists and

$$\sum_{n=1}^{\infty} [\| x^n - x^* \|^2 - \| x^{n-1} - x^* \|^2]_+ < +\infty,$$

Consequently, the sequence $\{x^n\}$ is bounded and

$$\lim_{n\to\infty} [\| x^n - x^* \|^2 - \| x^{n-1} - x^* \|^2]_+ = 0. \tag{20}$$

Thank to Lemma 3 (a) and (3), one has

$$\lim_{n\to\infty} \| w^n - x^n \|^2 = \lim_{n\to\infty} \alpha_n^2 \| x^n - x^{n-1} \|^2 \leq \lim_{n\to\infty} \alpha\alpha_n \| x^n - x^{n-1} \|^2 = 0. \tag{21}$$

So, $\{w^n\}$ is bounded since $\{x^n\}$ is bounded.

Theorem 5 *Let bifunction $f: H \times H \to \mathbb{R}$ satisfy the assumptions (M_1) - (M_4). Then, the sequence $\{x^n\}$ generated by Algorithm 1 converges weakly to a point $x^* \in Sol(EPs)$.*

Proof. Let $x^* \in Sol(EPs)$. By Lemma 3 (b), $\alpha_n \leq \alpha$ and (18), it follows that

$$\| x^{n+1} - x^* \|^2 \leq (1+\alpha_n) \| x^n - x^* \|^2 - \alpha_n \| x^{n-1} - x^* \|^2 + \alpha_n(1+\alpha_n)$$
$$\| x^n - x^{n-1} \|^2 - (\gamma + \kappa_n)(2 - \gamma - \kappa_n) \frac{(\xi_{n+1} - \nu\xi_n)^2}{(\xi_{n+1} + \nu\xi_n)^2} \| w^n - y^n \|^2$$

$$\leq \| x^n - x^* \|^2 + \alpha_n(\| x^n - x^* \|^2 - \| x^{n-1} - x^* \|^2) + \alpha_n(1+\alpha) \| x^n - x^{n-1} \|^2$$

$$-(\gamma + \kappa_n)(2 - \gamma - \kappa_n)\frac{(\xi_{n+1} - \nu\xi_n)^2}{(\xi_{n+1} + \nu\xi_n)^2} \| w^n - y^n \|^2$$

$$\leq \| x^n - x^* \|^2 + \alpha_n[\| x^n - x^* \|^2 - \| x^{n-1} - x^* \|^2]_+ + \alpha_n(1+\alpha) \| x^n - x^{n-1} \|^2$$

$$-(\gamma + \kappa_n)(2 - \gamma - \kappa_n)\frac{(\xi_{n+1} - \nu\xi_n)^2}{(\xi_{n+1} + \nu\xi_n)^2} \| w^n - y^n \|^2, \quad \forall k \geq 1.$$

Consequently

$$(\gamma + \kappa_n)(2 - \gamma - \kappa_n)\frac{(\xi_{n+1} - \nu\xi_n)^2}{(\xi_{n+1} + \nu\xi_n)^2} \| w^n - y^n \|^2$$

$$\leq \| x^n - x^* \|^2 - \| x^{n+1} - x^* \|^2 + \alpha_n[\| x^n - x^* \|^2 - \| x^{n-1} - x^* \|^2]_+$$
$$+\alpha_n(1+\alpha) \| x^n - x^{n-1} \|^2, \quad \forall k \geq 1. \tag{22}$$

Letting $k \to \infty$ in the above inequality and using Condition (1) Lemma 4, (22) and Lemma 3 (a), one has

$$\lim_{n\to\infty} \| w^n - y^n \| = 0.$$

Let $\Delta(x^n)$ denote the set of weak cluster points of the sequence $\{x^n\}$. We now show that $\Delta(x^n) \subset Sol(EPs)$. Indeed, let $\hat{x}$ be any point in $\Delta(x^n)$, then there exists a subsequence $\{x^{n_i}\}$ of $\{x^n\}$ converging weakly to $\hat{x}$. By Lemma 3 (a), the sequence $\{w^{n_i}\}$ converges weakly to $\hat{x}$. This together $\lim_{n\to\infty} \| w^n - y^n \| = 0$ implies that the sequence $\{y^{n_i}\}$ also converges weakly to $\hat{x}$. From $U_{y^n} \in B(U_{w^n}, L^* \| w^n - y^n \|)$, it follows that

$$\| U_{w^n} - U_{y^n} \| \leq L^* \| w^n - y^n \|,$$

and so $\lim_{n\to\infty} \| U_{w^n} - U_{y^n} \| = 0$. By (4), one has

$$\langle w^{n_i} - y^{n_i} - \xi_{n_i} U_{w^{n_i}}, y^{n_i} - x \rangle \geq 0, \ \forall x \in C.$$

Which together with $U_{y^{n_i}} \in \partial_2 f(y^{n_i}, y^{n_i})$ implies that

$$2\langle w^{n_i} - y^{n_i}, x - y^{n_i} \rangle \leq \xi_{n_i}\langle U_{w^{n_i}}, x - y^{n_i} \rangle$$
$$\leq \xi_{n_i}(\langle U_{y^{n_i}}, x - y^{n_i} \rangle + \langle U_{w^{n_i}} - U_{y^{n_i}}, x - y^{n_i} \rangle$$
$$\leq \xi_{n_i} f(y^{n_i}, x) + \xi_{n_i}\langle U_{w^{n_i}} - U_{y^{n_i}}, x - y^{n_i} \rangle.$$

From $\| w^n - y^n \| \to 0$ as $k \to \infty$ and $\{w^{n_i}\}$ is bounded, we have that $\{y^{n_i}\}$ also is bounded. For each fixed point $x \in C$ take the limit as $i \to \infty$ on the last inequality, using $\lim_{i\to\infty} \| w^{n_i} - y^{n_i} \| = 0$, $\lim_{n\to\infty} \| U_{w^{n_i}} - U_{y^{n_i}} \| = 0$, Condition (1) and the sequentially weakly upper semicontinuity of $f(\cdot, y)$, we get

$$f(\hat{x}, x) \geq 0 \quad \forall x \in C.$$

So, $\hat{x} \in Sol(EPs)$. By Lemma 10, the sequence $\{x^n\}$ generated by Algorithm 1 converges weakly to a point $x^* \in Sol(EPs)$.The theorem is proven.

Computational experiments

In this section, we introduce some numerical examples to illustrate proposed algorithms. All programming is coded in Matlab R2016a and the program was run on a PC Intel(R) Core(TM) i5-2430M CPU @ 2.40 GHz 4GB Ram. We used the Optimization Toolbox (fmincon) to solve strongly convex subproblems that are generated by proposed algorithms.

Example 1. Let $\mathrm{H} = \mathbb{R}^s$ and $C = \{x \in \mathbb{R}^s\colon\ \langle a, x\rangle = q\}$ $(0 \neq a \in \mathbb{R}^s,\ q \in \mathbb{R})$. Consider Problem (EPs) with the bifunction $f\colon \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}$ is defined by

$$2f(x,y) = \max\{\frac{1}{2} \| y \|^2 + q, \frac{1}{2} \| y \|^2 + \langle a, y\rangle\} - \max\{\frac{1}{2} \| x \|^2 + q, \frac{1}{2} \| x \|^2 + \langle a, x\rangle\}.$$

It is well-known that $h(x) = \max\{\frac{1}{2} \| x \|^2 + q, \frac{1}{2} \| x \|^2 + \langle a, x\rangle\}$ is convex, subdifferentiable on $\mathbb{R}^s$ and

$$\partial h(x) = \begin{cases} \{x + a\} & \text{if}\langle a, x\rangle > 0 \\ \{x\} & \text{if}\langle a, x\rangle > 0 \\ & \text{if}\langle a, x\rangle = q, \end{cases} \tag{23}$$

where $[x, x + a] = \{tx + (1 - t)(x + a)\colon t \in [0,1]\}$. It follows that

$$\rho(\partial_2 f(x,\cdot)(x), \partial_2 f(y,\cdot)(y)) = \| x - y \|,\ \forall x, y \in C.$$

It is easy to see that $f(x, y)$ satisfies assumptions (M_1)-(M_4).

Test 1. Let $n = 5$. We perform some experiments to show the numerical behaviors of Algorithm 3.1 for solving the Example 1. The initial points are $x^1 = x^0 = (-34,0,0,0,0)^\top$ and the data is chosen as follows: $a = (1,1,2,3,-1)^\top,\ q = -34,$

$$\xi_0 = 0.5, \qquad \gamma = 1, \qquad \alpha = 0.5, L^* = 2,$$
$$\kappa_n = \frac{1}{n+1}, \mu_n = \frac{1}{(n+1)^{1.5}}, \theta_n = \frac{1}{(2n+1)^{2.2}}.$$

Figure 1 shows convergent results of $x^n(i) - x^{n-1}(i)$, $i = 1,2,\cdots,5$, where $x^n(i)$ is the i-th coordinate of x^n.

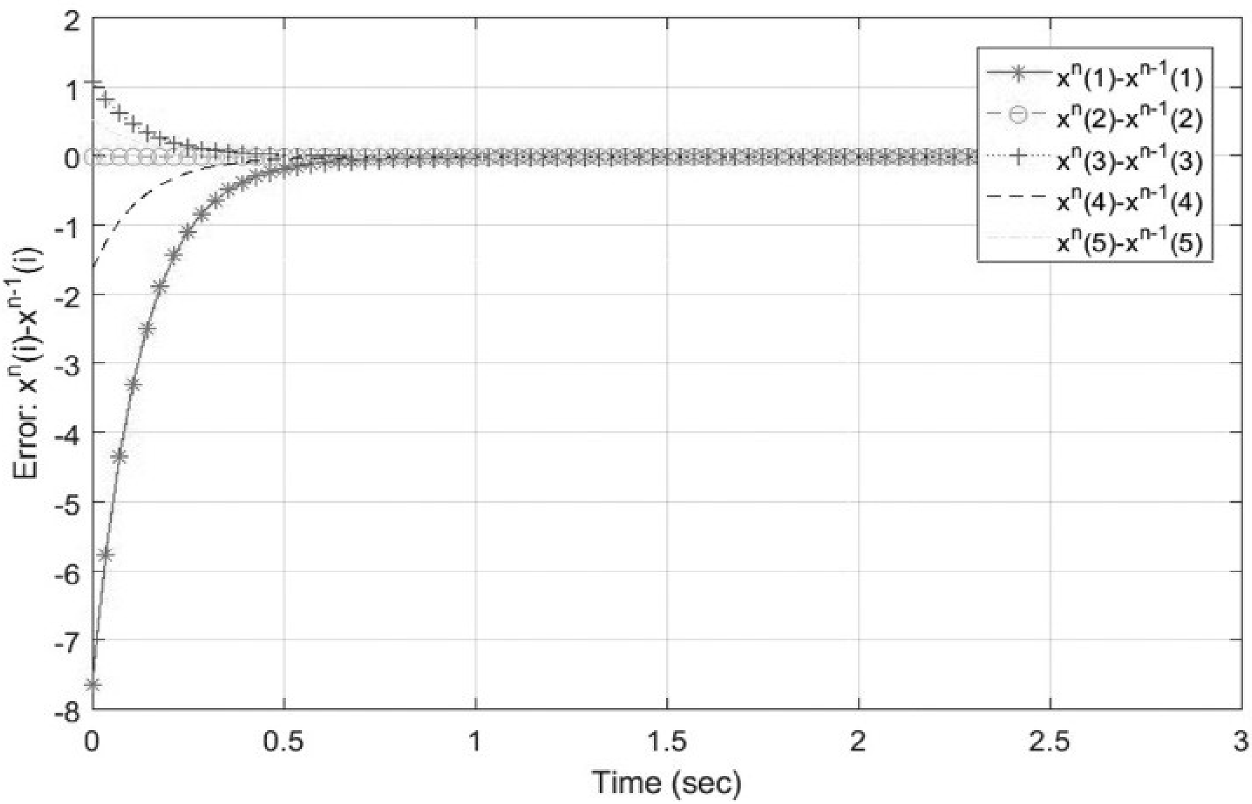

Figure 1. Convergence of Algorithm 3.1 with the tolerance 10^{-3}

Example 2. Let $\mathrm{H} = \mathbb{R}^s$, $b \in \mathbb{R}^m$, A be a $m \times s$ matrix. Consider Problem (EPs) with the feasible C is a polyhedral convex set given by

$$C = \{x \in \mathbb{R}^s : Ax \leq b\},$$

and the bifunction $f: \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}$ is defined as in (Van Thang, 2022):

$$f(x, y) = \langle Px + Qy + q, y - x \rangle.$$

where $q \in \mathbb{R}^n$, B, P and Q are $n \times n$ matrices such that Q is symmetric positive semidefinite and $P - Q$ is negative definite. As shown in (Konnov, 2001), the $f(x,y)$ satisfies the assumptions (M_1), (M_2) and (M_4). We have

$$f(x,y) + f(y,x) = -\langle (P-Q)(y-x), y-x \rangle < 0, \ \forall x \neq y,$$

which implies that $f(x,y)$ is strongly monotone on C. Since $f(\cdot,y)$ is upper semicontinuous, $f(x,\cdot)$ is convex and lower semicontinuous and $f(x,y)$ is strongly monotone, we have (EPs) has a unique solution ((Konnov, 2001), Proposition 2.1.16).

Test 2. Let $s = 5, m = 10$. In this test, we apply Algorithm 1 to solve Example 2 with different given initial points and parameters θ_n, κ_n. We will use $\xi_0 = 0.5$, $\gamma = 1$, $\alpha = 0.5, L^* = \| P - Q \| + 1$, $\mu_n = \frac{1}{(n+1)^{1.5}}$, for all k. The matrices P, Q, q are chosen as follows:

$$Q = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, P = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, q = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{pmatrix},$$

$$A^T = \begin{pmatrix} -1 & 1 & 2 & 0 & 1 & 1 & 1 & 3 & 2 & 2 \\ -1 & -2 & -1 & -2 & -1 & -2 & -3 & 3 & 4 & 2 \\ -1 & -1 & -0.5 & 1 & -1 & -2 & -4 & -3 & -5 & 2 \\ 0 & 2 & 1 & -2 & 1 & 1 & 2 & 2 & -3 & 1 \\ -1 & -0.5 & 2 & 1.5 & -2 & -1 & 3 & 2 & 5 & 0 \end{pmatrix},$$

$$b^T = (0,1,0,1,-1,2,2,-1,-1,-2).$$

The stopping criteria is $Err = \| x^n - x^{n-1} \| \leq \epsilon$ with $\epsilon = 10^{-3}$ and the approximate solution computed by Algorithm 3.1 is

$$x^* = (-1.9250, 0.3375, 0.5716, 0.4459, 1.2532)^T.$$

The computation results are shown in Table 1. From this table, we can make the following comments about the algorithm.

(a) The speed of our algorithm is less affected by the parameters θ_n and κ_n. This shows that the parameter ξ_n and τ_n is mostly updated based on previous iteration points.

(b) The program that encodes the proposed algorithm runs quickly if the initial point x^0 is close to a solution of the problem. Conversely, if the initial point x^0 is far from a solution then the program takes much more time.

Table 1. The comparative results for different starting points and parameters

	Parameters		Algorithm 1	
$Init. point\ x^1 = x^0$	κ_n	θ_n	Iterations.	CPU times
$(1,3,1,1,-2)^{\top}$	$\frac{1}{n+1}$	$\frac{1}{n^2+1}$	64	2.3967
$(1,3,1,1,-2)^{\top}$	$\frac{1}{2n+1}$	$\frac{1}{n^2+1}$	58	2.1043
$(1,3,1,1,-2)^{\top}$	$\frac{1}{3n+1}$	$\frac{1}{n^2+1}$	56	2.1237
$(1,3,1,1,-2)^{\top}$	$\frac{1}{4n+1}$	$\frac{1}{n^2+1}$	63	2.3043
$(1,3,1,1,-2)^{\top}$	$\frac{1}{5n+1}$	$\frac{1}{n^2+1}$	67	2.5329
$(1,3,1,1,-2)^{\top}$	$\frac{1}{n+1}$	$\frac{1}{n^4+1}$	60	1.9587
$(1,3,1,1,-2)^{\top}$	$\frac{1}{n+1}$	$\frac{1}{n^6+1}$	72	2.3572
$(1,3,1,1,-2)^{\top}$	$\frac{1}{n+1}$	$\frac{1}{n^8+1}$	67	2.4863
$(1,3,1,1,-2)^{\top}$	$\frac{1}{n+1}$	$\frac{1}{n^{10}+1}$	59	1.9983
$(2.4,0.6,1,0.25,1.3)^{\top}$	$\frac{1}{n+1}$	$\frac{1}{n^2+1}$	89	2.9828
$(4,6,5,3,7)^{\top}$	$\frac{1}{n+1}$	$\frac{1}{n^2+1}$	108	3.2384
$(7,8,6,6,13)^{\top}$	$\frac{1}{n+1}$	$\frac{1}{n^2+1}$	143	3.4251
$(11,13,12,21,24)^{\top}$	$\frac{1}{n+1}$	$\frac{1}{n^2+1}$	187	3.9448

Conclusions

We propose a new inertial subgradient projection algorithm for finding a solution of an equilibrium problem in a real Hilbert space. Our algorithm combines subgradient projection method and inertial techniques. Moreover, at each iteration, the self-adaptive is used. The convergent theorem are established under standard assumptions imposed on the equilibrium function. Several fundamental experiments are shown to illustrate our algorithm.

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