Determining Modes of the 2D g-Navier-Stokes Equations

Nguyen Dinh Thi[a*], Tran Quang Thinh[b]

Abstract:

The "*determining modes*" introduced by Prodi and Foias in 1967 say that if two solutions agree asymptotically in their P projection, then they are asymptotically in their entirety (see (Foias, 1967)). We study the initial boundary value problem for 2D g-Navier-Stokes (g-NVS) equations in bounded domains with homogeneous Dirichlet boundary conditions. We find an improved upper bound on the number of deterministic modes. Moreover, we slightly improve the estimate of the number of deterministic modes and achieve the upper limit of the Grashof Gr numerical order. These estimates are consistent with heuristic estimates based on physical arguments, extends previous results by O.P. Manley and Y.M. Treve (see (Foias, 1983)). The Gronwall lemma and Poincaré type inequality will play a central role in our computational technique as well as of the paper. Studying the properties of solutions is important to determine the behavior of solutions over a long period of time. The obtained result particularly extends previous results for 2D NVS equations.

Key words: *Navier-Stokes; weak solutions; determining modes; Grashof number; Dirichlet boundary conditions*

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[a] Faculty of Fundamental Sciences, Namdinh University of Technology Education; Phu Nghia Street, Loc Ha Ward, Nam Dinh City, Nam Dinh Province. e-mail: thispkt@gmail.com

[b] Faculty of Fundamental Sciences, Namdinh University of Technology Education; Phu Nghia Street, Loc Ha Ward, Nam Dinh City, Nam Dinh Province. e-mail: tqthinh.spktnd@moet.edu.vn

[*] Corresponding Author.

Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary Γ. We consider the following two-dimensional (2D) non-autonomous g -Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f(x,t) \text{ in } (0,T) \times \Omega, \\ \nabla \cdot (gu) = 0 \text{ in } (0,T) \times \Omega, \\ u = 0 \text{ on } (0,T) \times \Gamma, \\ u(x,0) = u_0(x),\ x \in \Omega, \end{cases} \tag{1}$$

where $u = u(x,t) = (u_1, u_2)$ is the unknown velocity vector, $p = p(x,t)$ is the unknown pressure, $\nu > 0$ is the kinematic viscosity coefficient, u_0 is the initial velocity.

The g-NVS equations are a variant model of the standard NVS equations. Furthermore, when $g \equiv \text{const}$ we have the usual NVS equations. The 2D g-NVS equations arise in a natural way when we study the standard 3D NVS equations in thin domains. We refer the reader to (Olson, 2008) for the 2D g-NVS equations from the 3D NVS equation and the relationship between them. As mentioned in (Olson, 2003), the good properties of the 2D g-NVS equations may lead to the beginning of the study of NVS equations on the thin 3D domain $\Omega_g = \Omega \times (0,g)$.

Recently, the existence of both weak and strong solutions to the 2D g-NVS equation has been investigated in (Foias, 1983, 1987). The existence of periodic solutions to the g-NVS equations has been investigated recently in (Foias, 1967). Furthermore, the long-term behavior of solutions in terms of the existence of universal, homogeneous and pullback attractors has been studied extensively in both autonomous and non-autonomous cases, see e.g. (Catania, 2012; Foias, 1984; Jones, 1993; Olson, 2003, 2008) and references therein. However, to the best of our knowledge, little is known about other properties of solutions to the 2D g-NVS equation. This is a driving force of the current paper.

We will study the number of ways to define 2D-Navier-Stokes in domains that are not necessarily bounded but satisfy the Poincaré inequality. In particular, the dependence of the determination methods of numbers on Grashof's numbers. To do this, we assume that the domain Ω and functions *f, g* satisfy the following hypotheses:

The domain Ω is an arbitrary (not necessarily bounded) domain of $\mathbb{R}^2$ satisfying the Poincaré inequality:

$$\int_\Omega \phi^2 g dx \le \frac{1}{\lambda_1} \int_\Omega |\nabla \phi|^2 \, g dx, \quad \text{for all } \phi \in C_0^\infty(\Omega); \tag{2}$$

where $\lambda_1 > 0$ is the first eigenvalue of the g-Stokes operator in Ω;

(F) $f \in L^2(0,T; H_g)$,

(G) $g \in W^{1,\infty}(\Omega)$ such that

$$0 < m_0 \le g(x) \le M_0 \text{ for all } x = (x_1, x_2) \text{ in } \Omega \text{ and } |\nabla g|_\infty^2 < m_0^2 \lambda_1. \quad (3)$$

The article structure is as follows. In Preliminaries, for convenience of the reader, we recall the preliminaries of the 2D g-NVS equations. As the main result of the paper, we show the number of modes defined in Determining modes.

Preliminaries

Let$\mathbb{L}^2(\Omega, g) = (L^2(\Omega, g))^2$ and $\mathbb{H}_0^1(\Omega, g) = (H_0^1(\Omega, g))^2$ be endowed, respectively, with the inner products

$$(u, v)_g = \int_\Omega u.v.g dx$$

where $u, v \in \mathbb{L}^2(\Omega, g)$

$$((u, v))_g = \int_\Omega \sum_{j=1}^2 \nabla u_j . \nabla v_j . g dx$$

where $u, v \in \mathbb{H}_0^1(\Omega, g)$,

with norms $|u|^2 = (u, u)_g$, $||u||^2 = ((u, u))_g$. From assumption $(\boldsymbol{G})$, the norms $|.|$ and $||.||$ are equivalent to the usual ones in $\mathbb{L}^2(\Omega, g)$ and in $\mathbb{H}_0^1(\Omega, g)$..

Let $\mathcal{V} = \{u \in (C_0^\infty(\Omega, g))^2 : \nabla \cdot (gu) = 0\}$.

We denote H_g as the closure of $\mathcal{V}$ in $\mathbb{L}^2(\Omega, g)$ and V_g as the closure of $\mathcal{V}$ in $\mathbb{H}_0^1(\Omega, g)$. Furthermore, $V_g \subset H_g \equiv H_g' \subset V_g'$ where the injections are continuous and dense. We use $||.||_*$ for the norm in V_g' and $\langle .,. \rangle$ for duality pairing between V_g andV_g'.

We define the trilinear b as follows

$$b_g(u, v, w) = \sum_{i,j=1}^2 \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j g dx,$$

whenever the integrals make sense.

It is easy to see that if $u, v, w \in V_g$, then $b_g(u, v, w) = -b_g(u, w, v)$.

Specially, $b_g(u, v, v) = 0$, $\forall u, v \in V_g$.

Set $A_g : V_g \to V_g'$ by $\langle A_g u, v \rangle = ((u, v))_g$, $B_g : V_g \times V_g \to V_g'$by $\langle B_g(u, v), w \rangle = b_g(u, v, w)$ and put $B_g u = B_g(u, u)$.

We denote $D(A_g) = \{u \in V_g : A_g u \in H_g\}$ then $D(A_g) = H^2(\Omega, g) \cap V_g$ and $A_g u = -P_g \Delta u, \forall u \in D(A_g)$, where P_g is the ortho-projector from $\mathbb{L}^2(\Omega, g)$ onto H_g.

Using the Ladyzhenskaya inequality and the Hölder inequality (when $n = 2$)

$|u|_{L^4} \le c|u|_g{}^{1/2}||u||_g{}^{1/2}, \ \ \forall u \in \mathbb{H}_0^1(\Omega, g).$

Lemma 1 (Özlük, 2016) When $n = 2$, then

$$|b_g(u,v,w)| \le \begin{cases} c_1|u|_g^{1/2}||u||_g^{1/2}||v||_g|w|_g^{1/2}||w||_g^{1/2}, & \forall u,v,w \in V_g, \\ c_2|u|_g^{1/2}||u||_g^{1/2}||v||_g^{1/2}|A_g v|_g^{1/2}|w|_g, & \forall u \in V_g, v \in D(A), w \in H_g, \\ c_3|u|_g^{1/2}|A_g u|_g^{1/2}||v||_g|w|_g, & \forall u \in D(A_g), v \in V_g, w \in H_g, \\ c_4|u|_g||v||_g|w|_g^{1/2}|A_g w|_g^{1/2}, & \forall u \in H_g, v \in V_g, w \in D(A_g), \end{cases} \tag{4}$$

an $|B_g(u,v)| + |B_g(v,u)| \le c_5||u||_g||v||_g^{1-\theta}|A_g v|_g^{\theta}, \quad \forall u \in V_g; v \in D(A_g),$ (5)

where $\theta \in (0,1)$; appropriate constant s $c_i with i = \overline{1,5}$. Moreover, For every $u, v \in D(A_g)$, then

$$|B_g(u,v)| \le c_6 \begin{cases} |A_g u|_g||v||_g, \\ ||u|_g||A_g v|_g. \end{cases} \tag{6}$$

where $u, v \in D(A_g)$.

Lemma 2. (Foias, 1987) Let $u \in \mathbb{L}^2(0,T;V_g)$, then the function $C_g u$ defined by

$$((C_g u(t), v)_g = \left(\left(\frac{\nabla g}{g}.\nabla\right)u, v\right)_g = b_g\left(\frac{\nabla g}{g}, u, v\right), \forall v \in V_g,$$

belongs to $\mathbb{L}^2(0,T;H_g)$ *and* $\mathbb{L}^2(0,T;V_g')$.

Moreover,

$$|C_g u(t)| \le \frac{|\nabla g|_\infty}{m_0}.||u(t)||_g, \text{ for } \forall\, t \in (0,T),$$

and

$$||C_g u(t)||_* \le \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}.||u(t)||_g, \text{ for } \forall\, t \in (0,T). \tag{7}$$

Since

$$-\frac{1}{g}(\nabla . g\nabla)u = -\Delta u - (\frac{\nabla g}{g}.\nabla)u,$$

we have

$$(-\Delta u, v)_g = ((u,v))_g + ((\frac{\nabla g}{g}.\nabla)u, v)_g = (A_g u, v)_g + ((\frac{\nabla g}{g}.\nabla)u, v)_g, \forall u, v \in V_g.$$

Lemma 3. (Jones, 1993)

Let ω be a locally integrable real valued function on $(0,\infty)$ satisfying the following conditions:

$$\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \omega\,(\tau)d\tau = \gamma > 0,$$

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \omega^-\,(\tau)d\tau = \Gamma < \infty,$$

where $\omega^- = max\{-\omega, 0\}$ and $0 < T < \infty$. Further, let ψ be a real valued locally integrable function defined on $(0,\infty)$ such that

$$\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \psi^+\,(\tau)d\tau = 0,$$

where $\psi^+ = max\{\psi, 0\}$. Suppose that γ is an absolutely continuous non-negative function on $(0,\infty)$ such that

$$\frac{d}{dt}\gamma + \omega\gamma \le \psi,$$

then $\gamma(t) \to 0$ when $t \to \infty$.

Using the above notations, we can rewrite the system (1) as abstract evolutionary equations

$$\begin{cases} \frac{du}{dt} + \nu A_g u + \nu C_g u + B_g(u,u) = f, \\ u = u_0. \end{cases} \tag{8}$$

Let

$$F = \limsup_{t\to\infty}\left(\int_\Omega |f(t,x)|^2 dx\right)^{1/2}.$$

We define the number Gr (generalized Grashof) $Gr = \frac{F}{\lambda_1 \nu^2}$.

The Gr number will act similarly to the Reynolds number and will be our branching parameter. In the next section, all our estimates will be based on the generalized Grahof number. Note that if f is time independent then Gr is Grashof's number $Gr = \frac{F}{\lambda_1 \nu^2}$.

Determining modes

The first m eigenfunctions of the $g - Stokes$ operator A_g. We define P_m orthogonal projections onto a linear space extending by $\{w_1, w_2, \ldots, w_m\}$ and $Q_m = I - P_m$.

Let u and v be weak solutions of the g-NVS equations, respectively.

$$\begin{cases} \dfrac{du}{dt} + \nu A_g u + \nu C_g u + B_g(u,u) = f_1, \\ u = u_0. \end{cases} \tag{9}$$

$$\begin{cases} \dfrac{dv}{dt} + \nu A_g v + \nu C_g v + B_g(v,v) = f_2, \\ v = v_0. \end{cases} \tag{10}$$

where $f_1, f_2 \in L^\infty(0,\infty; H_g)$.

A set of modes $\{w_j\}_{j=1}^m$ is called that determining if we have

$$\lim_{t\to\infty} |u(t) - v(t)|_g = 0,$$

whenever

$$\lim_{t\to\infty} |f_1(t) - f_2(t)|_g = 0,$$

and

$$\lim_{t\to\infty} |P_m u(t) - P_m v(t)|_g = 0.$$

Theorem 1. Assume that m satisfies

$$\frac{\lambda_{m+1}}{\lambda_1} \geq \frac{3^{1/2}}{\gamma_0} c_3 Gr,$$

where

$$\gamma_0 = \left(1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\right).$$

Then the number of determining modes is not greater than m. That is, if we have $\lim_{t\to\infty} \|P_m u(t) - P_m v(t)\|_g = 0,$ and $\lim_{t\to\infty} |f_1(t) - f_2(t)|_g = 0,$ then $\lim_{t\to\infty} \|u(t) - v(t)\|_g = 0.$ *Proof.*

Let $w = u - v$, $p = P_m w(t)$ and $q = Q_m w(t)$. Then, we assume that $|p| \to 0$ when

$t \to \infty$. Subtracting equation (10) from (9), we get

$$w_t + \nu A_g w + \nu C_g w + B_g(w,u) + B_g(u,w) - B_g(w,w) = f_1 - f_2. \tag{11}$$

Multiplying equation (11) by $A_g q$, we obtain

$$\begin{aligned} \frac{d}{dt}\|q\|_g^2 + 2\nu\left|A_g q\right|_g^2 + 2\nu(C_g w, A_g q)_g + 2(B_g(w,u) + B_g(u,w) - B_g(w,w), A_g q) \\ = 2(f_1 - f_2, A_g q)_g \,, \end{aligned} \tag{12}$$

Using Lemma 2 and (6), (8), we have

$$\begin{aligned} &\frac{d}{dt}\|q\|_g^2 + 2\nu\left|A_g q\right|_g^2 \\ &\qquad \le \frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\left|A_g q\right|_g^2 + 2(f_1 - f_2, A_g q)_g + 2\left|(B_g(w,w), A_g u)\right| \\ &\qquad + 2\left|(B_g(u,w) + B_g(w,u) - B\ \ (w,w), A_g p)\right|, \end{aligned}$$

or

$$\begin{aligned} &\frac{d}{dt}\|q\|_g^2 + 2\nu\gamma_0\left|A_g q\right|_g^2 \\ &\qquad \le 2(f_1 - f_2, A_g q)_g + 2\left|(B_g(w,w), A_g u)\right| \\ &\qquad + 2\left|(B_g(u,w) + B_g(w,u) - B\ \ (w,w), A_g p)\right|, \end{aligned}$$

Where

$$\gamma_0 = \left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right) > 0\,,$$

We use (4) and $|A_g^{3/2} p|_g \le \lambda_m^{3/2}|p|_g$ to obtain

$$\begin{aligned} &\left|(B_g(u,w) + B_g(w,u) - B_g(w,w), A_g p)\right| \\ &\qquad = \left|(B_g(u, A_g p), w) + (B_g(w, A_g p), u) - (B_g(w, A_g p), w)\right| \\ &\qquad \le (2c_1|u|_g^{1/2}\|u\|_g^{1/2}|w|_g^{1/2}\|w\|_g^{1/2} + c_1|w|_g\|w\|_g)\lambda_m^{3/2}|p|_g := M_1|p|_g\,, \end{aligned}$$

Since $||u||_g, ||v||_g$ remain bound when $t \to \infty$ and M_1 is bounded as $t \to \infty$.

We use the equation $B_g(w,w) = B_g(q,q) + B_g(p,w) + B_g(q,p)$ and (4) to obtain

$$\left|(B_g(w,w), A_g u)\right| \le \left|(B_g(q,q), A_g u)\right| + M_2|p|_g\left|A_g u\right|_g + c_3|p|_g\left|A_g q\right|_g\left|A_g u\right|_g\,,$$

where M_2 may be chosen to be $c_3\lambda_m^{1/2}(||u||_g + ||v||_g)$.

Using (4), we also have

$$(B_g(q,q), A_g u) \le c_3 |q|_g^{1/2} \left|A_g q\right|_g^{1/2} \|q\|_g \left|A_g u\right|_g \le \frac{c_3 \left|A_g q\right|_g}{\lambda_{m+1}^{1/2}} \|q\|_g \left|A_g u\right|_g.$$

Applying Cauchy's inequality, we conclude

$$\frac{d}{dt}\|q\|_g^2 + \|q\|_g^2 (\nu\gamma_0\lambda_{m+1} - \frac{3c_3^2 \left|A_g u\right|_g^2}{\nu\gamma_0\lambda_{m+1}}) \le \beta \,,$$

where

$$\beta = 2M_1|p|_g + 2M_2\left|A_g u\right|_g |p|_g + \frac{3}{\nu\gamma_0} c_3^2 \left|A_g u\right|_g^2 |p|_g^2 + \frac{3}{\nu\gamma_0}|f_1 - f_2|_g^2.$$

It follows the a priori estimate of the time average of $\left|A_g u\right|_g$

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} |A_g u|_g^2 d\tau \le \frac{F^2}{T\nu^3\lambda_1} + \frac{F^2}{\nu^2}\,,$$

for every $T > 0$ (here we take $T = (\nu\lambda_1)^{-1}$).

Applying Lemma 3 with

$$\gamma = \|q\|_g^2\,,$$

$$\omega = \nu\gamma_0\lambda_{m+1} - \frac{3c_3^2\left|A_g u\right|_g^2}{\nu\gamma_0\lambda_{m+1}}\,,$$

$$\psi = 2M_1|p|_g + 2M_2\left|A_g u\right|_g |p|_g + \frac{3}{\nu\gamma_0} c_3^2 \left|A_g u\right|_g^2 |p|_g^2 + \frac{3}{\nu\gamma_0}|f_1 - f_2|_g^2\,,$$

and noting that $\frac{\lambda_{m+1}}{\lambda_1} \ge \frac{3^{1/2}}{\gamma_0} c_3 Gr$, then the proof is complete.

Conclusion

In conclusion, we have presented an improved upper bound on the number of modes defined for the 2D g-NVS equations. Moreover, this is an important result in the study on the long time behavior of the solution when the time to infinity. The calculation techniques showed here can applied to other classes of equation systems such as Boussinesq and MHD.

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