# Capillary Waves at Cylindrical Interface of Two Immiscible Bose-Einstein Condensates

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**Abstract:**

By means of the hydrodynamic approach within the Gross-Pitaevskii (GP) theory, dispersion relation of Nambu-Goldstone modes at the interface of the system was found out. While component 1 of the system motion parallel to the interface the dispersion relations is of phonon and furthermore, the system becomes unstable for .

**Keywords:** *Hydrodynamic, Gross-Pitaevskii, interface, Nambu-Goldstone*

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**Introduction**

The theoretical studies of two immiscible BECs (E. Timmermans, 1998; P. Ao and S.T. Chui, 1998)) and the experimental realizations of such systems (C.J. Myatt, 1997; D.M. Stamper-Kurn, 1998; D.S. Hall, 1998; E.A. Cornell, 1998; Stenger, 1998) have allowed us to explore many interesting physical properties of BECs, in which the superfluid dynamics of interface between two segregated BECs has attracted special attention. Following this trend, in recent years one has focused on considerations of hydrodynamic instabilities at the interface of two BECs, such as the Kelvin-Helmholtz instability, the Rayleigh Taylor Instability and the Richtmyer-Meshkov instability (H. Takeuchi, 2010; K. Sasaki, 2011; A. Bezett, 2010). Combining the hydrodynamic approach and the Bogoliubov de Gennes method these considerations confirmed that the foregoing instabilities of fluid in classic hydro dynamics are also to take place for two segregated BECs. The present pap er deals with two-immiscible BECs with cylindrical configuration, in which the first component forms a cylinder along the z axis, which is surrounded by the second component. Such the system generates a cylindrical interface. It is known in fluid mechanics that a fluid cylinder will be unstable against break up into droplets if its length exceeds its circumference. This is the well-known capillary instability. As was indicated in Ref. (K. Sasaki, 2011) this is also the case for two segregated BECs. Our main aim is to investigate the capillary waves at this interface focusing on the exhibition of NG modes. To begin with, let us start from the Lagrangian of Gross-Pitaevskii theory

$$£ = \int d\vec{r}(P_1 + P_2 - g_{12}|\Psi_1|^2|\Psi_2|^2), \quad (1)$$

where

$$P_j = i\hbar\Psi^* j\frac{\partial\Psi_j}{\partial t} - \frac{\hbar^2}{2m_j}\left|\nabla\Psi_j\right|^2 - \frac{g_{ij}}{2}\left|\Psi_j\right|^4, \quad (2)$$

$\psi_j (j = 1, 2)$ are wave functions, $m_j$ atomic masses and the interaction coupling constants are defined as

$$g_{jk} = 2\pi\hbar^2 a_{jk}\left(m_j^{-1} + m_k^{-1}\right)g_{jk} = 2\pi\hbar^2 a_{jk}\left(m_j^{-1} + m_k^{-1}\right),$$

with $a_{jk}$ being the s-wave scattering length between the atoms in components $j$ and $k$. In the following we assume that

$$g_{12}^2 > g_{11}g_{22}g_{12}^2 > g_{11}g_{22},$$

implying that the two components are immiscible.

From Lagrangian (1) the Gross-Pitaevskii equation is deduced

$$i\hbar\frac{\partial\Psi_1}{\partial t}=\left(-\frac{\hbar^2}{2m_1}\nabla^2+g_{11}|\Psi_1|^2+g_{12}|\Psi_2|^2\right)\Psi_1.$$

$$i\hbar\frac{\partial\Psi_2}{\partial t}=\left(-\frac{\hbar^2}{2m_2}\nabla^2+g_{22}|\Psi_2|^2+g_{12}|\Psi_1|^2\right)\Psi_2, \quad (3)$$

It is clear that Lagrangian (1) and GP equation (3) are invariant under the transformations of the symmetry group $U(1)\times U(1)$. The presence of a cylindrical interface will spontaneously break down the translations in the (x, y)-plane. Thus, the system possesses four broken symmetries. In Lorentz invariant systems the Goldstone states that the number of NG modes coincides with the number of broken continuous symmetries. However, this statement fails for our system being not Lorentz invariant.

The paper is organized as follows. Section 2 is devoted to detailed investigation of physical phenomena occurring at a cylindrical interface of two immiscible BECs at rest and in motion. The conclusion and discussion are presented in Section 3.

**Laplace Equation and NG modes**

For cylindrical configuration we assume the first (second) component is inside (outside) the cylinder with radius R and the interface between them is located at $r=R(z,t)$, neglecting thickness. Then the Lagrangian (1) is approximated by

$$£=2\pi\int dz\left(\int_0^R rdrP_1+\int_R^\infty rdrP_2\right)-\alpha S,$$

where $\alpha$ is the interface tension (B. Van Schaeybroeck, 2008) and S the area of interface. From the foregoing equation the Laplace equation (L.D. Landau, 1987) is derived

$$P_1(R,z,t)-P_2(R,z,t)=\alpha\left(\frac{1}{R_1}+\frac{1}{R_2}\right), \quad (4)$$

in which $R_1$ and $R_2$ are the principal radii of the interface curvature. To explore the Laplace equation (4) let us at first determine the wave functions $\psi_j(r, z, t)$ in the linear approximation employed by (K. Sasaki, 2011). We write them as

$$\Psi_j(r,z,t)=\sqrt{n_j(r,z,t)}.\ e^{i\Phi j(r,z,t)}. \quad (5)$$

In which

$$n_j(r,z,t)=n_{j0}+\delta n_j(r,z,t),$$

$$\Psi_j(r,z,t)=-\frac{g_{jj}n_{j0}}{\hbar}+\delta\Phi_j. \quad (6)$$

Substituting (5) and (6) into the Gross-Pitaevskii equation (3) and keeping only the first order of $\delta\varphi_j$, $\delta n_j$ we are led to the equations

$$\nabla\overrightarrow{v_j} = 0, \quad \text{(7a)}$$

$$h\frac{\partial}{\partial t}\left(\delta\Phi_j\right) + g_{\text{ii}}\delta\text{n}_j = 0, \quad \text{(7b)}$$

with the velocity

$$\overrightarrow{v_j} = \frac{\hbar}{m_j}\vec{\nabla}\delta\Phi_j. \quad \text{(8)}$$

Eq.(7a) means that in the approximation the fluid turns out to be incompressible. Next, we adopt the ansatz

$$\delta\Phi_j(r,z,t) = R_j(r)\chi_j(\sigma),\ \sigma = kz - \omega t. \quad \text{(9)}$$

Taking into account (7b) and (9) Eq. (7a) leads to

$$\left(\frac{d^2}{dz^2} + k^2\right)\chi_j = 0, \quad \text{(10a)}$$

$$\frac{d^2R_j}{dz^2} + \frac{1}{r}\frac{dR_j}{dr} - k^2R_j = 0. \quad \text{(10b)}$$

Assuming that $k_2 > 0$, Eq.(10a) possesses the general solutions

$$\chi_j(\sigma) = A_j cos\sigma + B_j sin\sigma, \quad \text{(11)}$$

and the solutions to Eq. (10b) are expressed through the modified Bessel functions of the first and second kind

$$R_1(r) = const.\, I_0(kr),$$
$$R_2(r) = const.\, K_0(kr). \quad \text{(12)}$$

Combining (11) and (12) the general solutions of the system (10) read

$$\delta\Phi_1 = I_0(kr)[A_1 cos\sigma + B_1 sin\sigma],$$

$$\delta\Phi_2 = K_0(kr)[A_2 cos\sigma + B_2 sin\sigma],$$

which for simplicity are chosen as

$$\delta\Phi_1 = A_1 I_0(kr)cos\sigma,$$
$$\delta\Phi_2 = A_2 I_0(kr)cos\sigma. \quad \text{(13)}$$

Inserting (12) into (7b) yields

$$g_{11}\delta n_1 = -A_1\hbar\omega I_0(kr)sin\sigma,$$
$$g_{22}\delta n_2 = -2\hbar\omega K_0(kr)sin\sigma. \quad \text{(14)}$$

Taking into account (6), (13) and (14) the expression (5) takes the form

$$\psi_1 = \sqrt{n_{10} - \frac{A_1 \hbar\omega}{g_{11}} I_0(kr) sin\sigma} \,.e^{i\left(-\frac{g_{11}n_{10}}{\hbar}t + A_1 I_0(kr)cos\sigma\right)},$$

$$\psi_2 = \sqrt{n_{20} - \frac{A_2 \hbar\omega}{g_{22}} K_0(kr) sin\sigma} \,.e^{i\left(-\frac{g_{22}n_{20}}{\hbar}t + A_2 K_0(kr)cos\sigma\right)}.$$

So far, the analytical expressions for wave functions were established in the linear approximation. They will be used to determine the shape of interface and phonon dispersion relation. To do this, we assume the boundary condition

$$\frac{\partial R}{\partial t} = \frac{\hbar}{m_1}\left(\frac{\partial \delta \Phi_1}{\partial r}\right)_{r=R_0} = \frac{\hbar}{m_2}\left(\frac{\partial \delta \Phi_2}{\partial r}\right)_{r=R_0}, \qquad (15)$$

$$\omega \frac{d\vec{R}(\sigma)}{d\sigma} = -A_1 \frac{\hbar K}{m_1} I_1(kR_0) cos\sigma = -A_2 \frac{\hbar K}{m_2} K_1(kR_0) cos\sigma,$$

whose solution is straightforwardly derived

$$\bar{R}(\sigma) = \varepsilon sin\sigma, \qquad (16)$$

with $|\varepsilon| \ll 1|\varepsilon| \ll 1$ and $\omega$ describing the phonon dispersion relation

$$\omega = -\hbar K \frac{A_1 I_1(kR_0)}{\varepsilon m_1} = \hbar K \frac{A_2 K_1(kR_0)}{\varepsilon m_2}, \qquad (17)$$

which exhibits the quantum character of phonon. For convenience we take $\varepsilon > 0$ then the condensates are super fluids for $A_1 < 0$, $A_2 > 0$ and the speed of sound reads

$$c = -\hbar \frac{A_1 I_1(kR_0)}{\varepsilon m_1} = \hbar \frac{A_2 K_1(kR_0)}{\varepsilon m_2} > 0.$$

Imposing

$$\delta n_j = \delta \overline{n_j}(r) sin\sigma,$$

and taking into account (14), (17) we arrive at

$$g_{11}\delta\bar{n}_1(r) = \frac{\varepsilon\omega^2 m_1}{kI_1(kR_0)} I_0(kr).$$

$$g_{22}\delta\bar{n}_2(r) = -\frac{\varepsilon\omega^2 m_2}{kK_1(kR_0)} K_0(kr), \qquad (18)$$

$$\omega^2 = \frac{\alpha k(k^2R_0^2 - 1)}{R_0^2\left(m_1 n_{10}\frac{I_0(kR_0)}{I_1(kR_0)} + m_2 n_{20}\frac{K_0(kR_0)}{K_1(kR_0)}\right)}. \qquad (19)$$

Ultimately, substitution of (6), (13), (14) and (18) into the Laplace equation (4) yields the dispersion relation for cylindrical interface

From Eq (19) we have $\omega \sim k^{3/2}$, this is the dispersion relations is of ripplon. The quantum-mechanical pressure is included in the interfacial tension coefficient$\alpha\alpha$and Eq. (19) does not contain any explicit quantum correction term.

For $0 < k^2 R_0^2 < 1$, the right-hand side of Eq. (19) is negative and the frequency $\omega$ is pure imaginary. The mode with a wavelength larger than$2\pi R_0$ is therefore dynamically unstable.

Now let us extend to the case when the first component flows along the positive direction of the z axis with velocity $V$, while the second one is at rest. Then the corresponding stationary state takes the form

$$\Phi_1 = -\frac{g_{11} n_{10}}{h} t + \frac{m_1}{h} V z + \delta\Phi_1,$$

$$\Phi_2 = -\frac{g_{22} n_{20}}{h} t + \delta\Phi_2,$$

with $\delta\Phi_1$ given in (13). For the moving case the boundary condition (15) is modified to be

$$\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial z}\right) R(\sigma) = \frac{h}{m_1}\left(\frac{\partial \delta\Phi_1}{\partial r}\right)_{r=R_0},$$
$$\frac{\partial}{\partial t} R(\sigma) = \frac{h}{m_2}\left(\frac{\partial \delta\Phi_2}{\partial r}\right)_{r=R_0}, \qquad (20)$$

where $R(\sigma)$ denotes the equation of the cylindrical interface:

$$R(\sigma) = R_0 + \bar{R}(\sigma), \qquad |\bar{R}(\sigma)| \ll 1 \,. \qquad (21)$$

Inserting (21) into (20) gives

$$(-\omega + Vk)\frac{d\bar{R}(\sigma)}{d\sigma} = \frac{hk}{m_1} A_1 I_1(kR_0) cos\sigma,$$

$$\omega \frac{d\bar{R}(\sigma)}{d\sigma} = \frac{hk}{m_2} A_2 K_1(kR_0) cos\sigma.$$

Therefrom we obtain the solution representing the interface (21)

$$\bar{R}(\sigma) = \varepsilon sin\sigma \,, \qquad (22)$$

with $|\varepsilon| \ll 1$ and $\omega$ satisfying the dispersion relation of two phonon attributing to two condensates

$$\omega = \left(V - \frac{\hbar}{\varepsilon m_1} A_1 I_1(kR_0)\right) k \sim k,$$

$$\omega = \frac{\hbar k}{\varepsilon m_2} A_2 K_1(kR_0) \sim k. \tag{23}$$

The corresponding speeds of sound read

$$c_1 = V - \frac{\hbar}{\varepsilon m_1} A_1 I_1(kR_0) > 0,$$

$$c_2 = \frac{\hbar}{\varepsilon m_2} A_2 K_1(kR_0) > 0,$$

implying that the Landau instability no longer takes place.

In favor of (23) it is evident that

$$g_{11}\delta\bar{n}_1(r) = \frac{\varepsilon m_1(\omega - Vk)\omega}{kI_1(kR_0)} I_0(kr),$$

$$g_{22}\delta\bar{n}_2(r) = -\frac{\varepsilon m_2 \omega^2}{kK_1(kR_0)} K_0(kr). \tag{24}$$

Finally, substituting (6), (13) and (24) into the Laplace equation (4) we arrive at the equation for frequency

$$\left(\frac{m_1 n_{10} I_0(kR_0)}{kI_1(kR_0)} + \frac{m_2 n_{20} K_0(kR_0)}{kK_1(kR_0)}\right)\omega^2 + \frac{m_1 n_{10} I_0(kR_0)V}{I_1(kR_0)}\omega + \frac{\alpha}{R_0^2}(1 - k^2 R_0^2) = 0. \tag{25}$$

Eq. (25) provides two modes

$$\omega = \frac{k\left(-BV \pm \sqrt{B^2V^2 + \frac{4A\alpha}{kR_0^2}\left(k^2{R_0}^2 - 1\right)}\right)}{2A}. \tag{26}$$

here

$$A = \frac{m_1 n_{10} I_0(kR_0)}{I_1(kR_0)} + \frac{m_2 n_{20} K_0(kR_0)}{K_1(kR_0)},$$

$$B = \frac{m_1 n_{10} I_0(kR_0)}{I_1(kR_0)}.$$

From Eq. (26) we see with $0 < k^2 R_0^2 < 1$, the frequency $\omega$ is negative. Therefore, there is instability at the interface between the two BEC components.

## Conclusion and discussion

In this paper, we obtain the following main results:

The refractive index formula of the NG has been discovered at the interface of the two components in the cylindrical space when both components are stationary and even when one component is in motion. From the obtained results, we also observe that the instability phenomenon at the interface occurs when $0 < k^2 R_0^2 < 1$.

When component 1 of the system moves parallel to the interface, the NG mode changes from a complex refractive index formula to a phonon mode, but the occurrence of instability at the interface is not dependent on the velocity of the system.

Comparing with (B. Van Schaeybroeck, 2008; Joseph O. Indekeu, 2015; H. Takeuchi, 2013), we find that the NG mode not only depends on boundary conditions but also the spatial confinement is dependent on the shape of the interface.

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